A Note on the Budgeted Maximization of Submodular Functions

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Abstract

Many set functions F in combinatorial optimization satisfy the diminishing returns property $F(\mathcal{A} \cup \{X\}) - F(A) \geq F(\mathcal{A}' \cup \{X\}) - F(\mathcal{A}')$ for $\mathcal{A} \subset \mathcal{A}'$. Such functions are called *submodular*. A result from Nemhauser et.al. states that the problem of selecting k-element subsets maximizing a nondecreasing submodular function can be approximated with a constant factor $(1-1/e)$ performance guarantee. Khuller et.al. showed that for the special submodular function involved in the MAX-COVER problem, this approximation result generalizes to a budgeted setting under linear nonnegative cost-functions. They proved a $(1 - 1/\sqrt{e})$ approximation guarantee for a modified Greedy algorithm, and show how a $(1 - 1/e)$ guarantee can be achieved using partial enumeration. In this note, we extend their results to general submodular functions. Motivated by the problem of maximizing entropy in discrete graphical models, where the submodular objective cannot be evaluated exactly, we generalize our result to account for absolute errors.

1 Introduction

Many set functions F in combinatorial optimization satisfy the diminishing returns property $F(\mathcal{A} \cup X) - F(A) \geq F(\mathcal{A}' \cup X) - F(\mathcal{A}')$ for $\mathcal{A} \subset \mathcal{A}'$, i.e. adding an element to a smaller set helps more than adding it to a larger set. Such functions are called *submodular*. Here and in the following, we write $\mathcal{A} \cup X$ instead of $\mathcal{A} \cup \{X\}$ to simplify notation. The submodular function motivating our research is the joint entropy $H(\mathcal{A})$ for a set of random variables $\mathcal{A}$. The entropy of a distribution $P : \{x_1, \ldots, x_d\} \rightarrow [0,1]$ is defined as

$$H(P) = -\sum_k P(x_k) \log P(x_k),$$

measuring the number of bits required to encode $\{x_1, \ldots, x_d\}$ [1]. If $\mathcal{A}$ is a set of discrete random variables $\mathcal{A} = \{X_1, \ldots, X_n\}$, then their entropy $H(\mathcal{A})$ is defined as the entropy of their joint distribution. The conditional entropy $H(\mathcal{A} \mid \mathcal{B})$ for two subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{V}$ is defined as

$$H(\mathcal{A} \mid \mathcal{B}) = -\sum_{\substack{\mathbf{a} \in \mathrm{dom}\,\mathcal{A} \\ \mathbf{b} \in \mathrm{dom}\,\mathcal{B}}} P(\mathbf{a}, \mathbf{b}) \log P(\mathbf{a} \mid \mathbf{b}),$$

measuring the expected uncertainty about variables $\mathcal{A}$ after variables $\mathcal{B}$ are observed.

Using the chain-rule of entropies [1], $H(\mathcal{A} \cup \mathcal{B}) = H(\mathcal{A} \mid \mathcal{B}) + H(\mathcal{B})$, we can compute $H(\mathcal{A} \cup X) - H(\mathcal{A}) = H(X|\mathcal{A})$. The *information never hurts* principle [1], $H(X \mid \mathcal{A}) \geq H(X \mid \mathcal{A}')$ for all $\mathcal{A} \subseteq \mathcal{A}'$, proves submodularity of the entropy. In the discrete setting, $H(X \mid \mathcal{A})$ is also always non-negative, hence the entropy is nondecreasing.

In practice, a commonly used algorithm for selecting a set of variables with maximum entropy is to greedily select the next variable to observe as the most uncertain variable given the ones observed thus far:

$$X_k := \operatorname*{argmax}_X H(X \mid \{X_1, \ldots X_{k-1}\}), \tag{1.1}$$

which is again motivated by the chain-rule.

It is no surprise that this problem has been tackled with heuristic approaches, since even the unit cost has been shown to be **NP**-hard for multivariate Gaussian distributions [3], and a related formulation has been shown to be **NP**$^{\mathbf{PP}}$-hard even for discrete distributions that can be represented by polytree graphical models [4].

Fortunately, a result from Nemhauser et.al. [5] states that the problem of selecting k-element subsets maximizing a nondecreasing submodular function can be approximated with a constant factor $(1-1/e)$ performance guarantee, using the greedy algorithm as mentioned above. Khuller et.al. [2] showed that for the special submodular function involved in the MAX-COVER problem, this approximation result generalizes to a budgeted setting under linear nonnegative cost-functions. They proved a $(1 - 1/\sqrt{e})$ approximation guarantee for a modified Greedy algorithm, and show how a $(1 - 1/e)$ guarantee can be achieved using partial enumeration. In this note, we extend their results to general submodular functions. A different proof of the $(1-1/e)$ approximation guarantee for the partial enumeration algorithm has previously been presented by Sviridenko [6].

Motivated by the problem of maximizing entropy in discrete graphical models, where the conditional entropies in (1.1) can in general not be evaluated both exactly and efficiently [4], we generalize our result to account for absolute errors. Our derivations in the following sections closely follows the analysis presented in [2].

2 Budgeted maximization of submodular functions

Let $\mathcal{V}$ be a finite set, and $F : \mathcal{V} \rightarrow \mathbb{R}$ be a set function with $F(\emptyset) = 0$. F is called *submodular* if $F(\mathcal{A} \cup X) - F(\mathcal{A}) \geq F(\mathcal{A}' \cup X) - F(\mathcal{A}')$ for all $\mathcal{A} \subset \mathcal{V}$ and $X \in \mathcal{V} \setminus \mathcal{A}$. F is called non-decreasing if $F(\mathcal{A} \cup X) - F(\mathcal{A}) \geq 0$ for all $\mathcal{A} \subset \mathcal{V}$ and $X \in \mathcal{V} \setminus \mathcal{A}$. The quantities $F'(\mathcal{A}; X) := F(\mathcal{A} \cup X) - F(\mathcal{A})$ are called *marginal increases* of F with respect to $\mathcal{A}$ and X. Furthermore define a cost function $c : \mathcal{V} \rightarrow \mathbb{R}^+$, associating a positive cost $c(X)$ with each element $X \in \mathcal{V}$. We extend c linearly to sets: For $\mathcal{A} \subset \mathcal{V}$ define

$$c(\mathcal{A}) = \sum_{X \in \mathcal{A}} c(X).$$

For a budget $B > 0$, the budgeted maximization problem is to maximize

$$OPT = \underset{\mathcal{A} \subset \mathcal{V} : c(\mathcal{A}) \leq B}{\operatorname{argmax}} F(\mathcal{A}) \tag{2.1}$$

Note that the exclusion of zero cost does not incur loss of generality because since the submodular functions are nondecreasing. In addition, without loss of generality we assume that $c(X) \leq B$ for all $X \in \mathcal{V}$. We refer to $c(\mathcal{A}) = |\mathcal{A}|$ as the unit-cost case.

3   A constant factor approximation

In analogy to the unit-cost case discussed in [5], we analyze the greedy algorithm, where the greedy rule adds to set $\mathcal{A}$ the element X^* such that

$$X^* = \max_{X \in \mathcal{W} \setminus \mathcal{G}_{i-1}} \frac{\hat{F}'(\mathcal{G}_{i-1}; X)}{c(X_i)}.$$

Khuller et.al. [2] prove that the simple greedy algorithm with this greedy selection rule has unbounded approximation ratio. They suggest a small modification, considering the best single element solution as alternative to the output of the naive greedy heuristic, which, as they prove, guarantees a constant factor approximation for the budgeted MAX-k-COVER problem. Their algorithm is stated here as Algorithm 1, and we extend their analysis to the case of general submodular functions. Motivated by the entropy maximization problem where we cannot efficiently evaluate the marginal increases $F'(\mathcal{A}; X)$ exactly [4], we only assume that we can evaluate $\hat{F}'(\mathcal{A}; X)$ such that $\left| \hat{F}'(\mathcal{A}; X) - F'(\mathcal{A}; X) \right| \leq \varepsilon$ for some $\varepsilon > 0$.

Input: $d > 0, B > 0, \mathcal{W} \subseteq \mathcal{V}$
Output: Selection $\mathcal{A} \subseteq \mathcal{W}$
begin
    $\mathcal{A}_1 := \operatorname{argmax}\{F(\{X\}) : X \in \mathcal{W}, c(X) \leq B\};$
    $\mathcal{A}_2 := \emptyset;$
    $\mathcal{W}' := \mathcal{W};$
    while $\mathcal{W}' \neq \emptyset$ **do**
        foreach $X \in \mathcal{W}'$ **do** $\Delta_X := \hat{F}'(\mathcal{A}_2; X);$
        $X^* := \operatorname{argmax}\{\Delta_X / c(X) : X \in \mathcal{W}'\};$
1   **if** $c(\mathcal{A}_2) + c(X^*) \leq B$ **then** $\mathcal{A}_2 := \mathcal{A}_2 \cup X^*;$
        $\mathcal{W}' := \mathcal{W}' \setminus X^*;$
    end
    return $\underset{\mathcal{A} \in \{\mathcal{A}_1, \mathcal{A}_2\}}{\operatorname{argmax}} F(\mathcal{A})$
end

Algorithm 1: Approximation algorithm for budgeted case.

Let us consider the computation of the set $\mathcal{A}_2$ in Algorithm 1. Renumber $\mathcal{V} = \{X_1, \ldots, X_n\}$ and define $\mathcal{G}_0 = \emptyset$ and $\mathcal{G}_i = \{X_1, \ldots, X_i\}$ such that

$$\frac{F(\mathcal{G}_i) - F(\mathcal{G}_{i-1}) + \varepsilon}{c(X_i)} \geq \max_Y \frac{F(\mathcal{G}_{i-1} \cup Y) - F(\mathcal{G}) - \varepsilon}{c(Y)}.$$

The sequence $(G_j)_j$ is motivated by the simple greedy rule, adding, for a prior selection $\mathcal{G}_{i-1}$, the element X_i such that

$$X_i = \max_{X \in \mathcal{W} \setminus \mathcal{G}_{i-1}} \frac{\hat{F}'(\mathcal{G}_{i-1}; X)}{c(X_i)}.$$

Let $l = \max\{i : c(\mathcal{G}_i) \leq B\}$ be the largest integer such that the if-condition in Line 1 is always satisfied for the first l iterations of Algorithm 1. Hence $\mathcal{A}_2 \supseteq \mathcal{G}_l$. Let $c_{min} = \min_X c(X)$. We first prove the following Theorem:

Theorem 1 (adapted from [2]). *Algorithm 1 achieves an*

$$\left(1 - 1/\sqrt{e}\right) F(OPT) - \frac{2B\varepsilon}{c_{min}}$$

approximation for (2.1), *using* $\mathcal{O}(|\mathcal{W}|^2)$ *evaluations of* $\hat{F}'$.

To prove Theorem 1, we need two lemmas. Let $w = |OPT|$.

Lemma 2 (generalized from [2]). *For* $i = 1, \ldots, l+1$, *it holds that*

$$F(\mathcal{G}_i) - F(\mathcal{G}_{i-1}) \geq \frac{c(X_i)}{B}(F(OPT) - F(\mathcal{G}_{i-1})) - \varepsilon\left(1 + \frac{wc(X_i)}{B}\right)$$

Proof. Using monotonicity of F, we have

$$F(OPT) - F(\mathcal{G}_{i-1}) \leq F(OPT \cup \mathcal{G}_{i-1}) - F(\mathcal{G}_{i-1}) = F(OPT \setminus \mathcal{G}_{i-1} \cup \mathcal{G}_{i-1}) - F(\mathcal{G}_{i-1})$$

Assume $OPT \setminus \mathcal{G}_{i-1} = \{Y_1, \ldots, Y_m\}$, and let for $j = 1, \ldots, m$

$$Z_j = F(\mathcal{G}_{i-1} \cup \{Y_1, \ldots, Y_j\}) - F(\mathcal{G}_{i-1} \cup \{Y_1, \ldots, Y_{j-1}\}).$$

Then $F(OPT) - F(\mathcal{G}_{i-1}) \leq \sum_{j=1}^{m} Z_j$.

Now notice that

$$\frac{Z_j - \varepsilon}{c(Y_j)} \leq \frac{F(\mathcal{G}_{i-1} \cup Y_j) - F(\mathcal{G}_{i-1}) - \varepsilon}{c(Y_j)} \leq \frac{F(\mathcal{G}_i) - F(\mathcal{G}_{i-1}) + \varepsilon}{c(X_i)}$$

using submodularity in the first and the greedy rule in the second inequality. Since $\sum_{j=1}^{m} c(Y_j) \leq B$ it holds that

$$F(OPT) - F(\mathcal{G}_{i-1}) = \sum_{j=1}^{m} Z_j \leq B\frac{F(\mathcal{G}_i) - F(\mathcal{G}_{i-1}) + \varepsilon}{c(X_i)} + m\varepsilon \leq B\frac{F(\mathcal{G}_i) - F(\mathcal{G}_{i-1}) + \varepsilon}{c(X_i)} + w\varepsilon$$

$\square$

Lemma 3 (adapted from [2]). *For* $i = 1, \ldots, l+1$ *it holds that*

$$F(\mathcal{G}_i) \geq \left[1 - \prod_{k=1}^{i}\left(1 - \frac{c(X_k)}{B}\right)\right] F(OPT) - \left(\frac{B}{c(X_i)} + w\right)\varepsilon.$$

Proof. Let $i = 1$ for sake of induction. We need to prove that $F(\mathcal{G}_1) \geq \frac{c(X_1)}{B}F(OPT) - \left(\frac{B}{c(X_i)} + w\right)\varepsilon$. This follows from Lemma 2 and since

$$\frac{B}{c(X_i)} + w \geq 1 + \frac{wc(X_i)}{B}.$$

Now let $i > 1$. We have

$$F(\mathcal{G}_i) = F(\mathcal{G}_{i-1}) + [F(\mathcal{G}_i) - F(\mathcal{G}_{i-1})]$$

$$\geq F(\mathcal{G}_{i-1}) + \frac{c(X_i)}{B}[F(OPT) - F(\mathcal{G}_{i-1})] - \varepsilon\left(1 + \frac{wc(X_i)}{B}\right)$$

$$= \left(1 - \frac{c(X_i)}{B}\right)F(\mathcal{G}_{i-1}) + \frac{c(X_i)}{B}F(OPT) - \varepsilon\left(1 + \frac{wc(X_i)}{B}\right)$$

$$\geq \left(1 - \frac{c(X_i)}{B}\right)\left[\left(1 - \prod_{k=1}^{i-1}\left(1 - \frac{c(X_k)}{B}\right)\right)F(OPT) - \left(\frac{B}{c(X_i)} + w\right)\varepsilon\right] + \frac{c(X_i)}{B}F(OPT) - \varepsilon\left(1 + \frac{wc(X_i)}{B}\right)$$

$$= \left(1 - \prod_{k=1}^{i}\left(1 - \frac{c(X_k)}{B}\right)\right)F(OPT) - \varepsilon\left(1 + \frac{wc(X_i)}{B}\right) - \left(\frac{B}{c(X_i)} + w\right)\varepsilon\left(1 - \frac{c(X_i)}{B}\right)$$

$$= \left(1 - \prod_{k=1}^{i}\left(1 - \frac{c(X_k)}{B}\right)\right)F(OPT) - \left(\frac{B}{c(X_i)} + w\right)\varepsilon$$

using Lemma 2 in the first and the induction hypothesis in the second inequality. $\square$

From now on let $\beta = \frac{2B}{c_{min}} \geq \frac{B}{c_{min}} + w$.

Proof of Theorem 1. As observed by [2], it holds that if $a_1, \ldots, a_n \in \mathbb{R}^+$ such that $\sum a_i = \alpha A$ and $\alpha \in (0, 1]$, the function $(1 - \prod_{i=1}^{n}(1 - \frac{a_i}{A}))$ achieves its minimum at $a_1 = \cdots = a_n = \frac{\alpha A}{n}$. We have

$$
\begin{aligned}
F(\mathcal{G}_{l+1}) &\geq \left[1 - \prod_{k=1}^{l+1}\left(1 - \frac{c(X_k)}{B}\right)\right] F(OPT) - \beta\varepsilon \\
&\geq \left[1 - \prod_{k=1}^{i}\left(1 - \frac{c(X_k)}{c(\mathcal{G}_{l+1})}\right)\right] F(OPT) - \beta\varepsilon \\
&\geq \left[1 - \left(1 - \frac{1}{l+1}\right)^{l+1}\right] F(OPT) - \beta\varepsilon \\
&\geq \left(1 - \frac{1}{e}\right) F(OPT) - \beta\varepsilon
\end{aligned}
$$

where the first inequality follows from Lemma 3 the second inequality follows from the fact that $c(\mathcal{G}_{l+1}) > B$ since it violates the budget, and the third inequality follows from above observation with $\alpha = 1$.

Furthermore note, that the violating increase $F(\mathcal{G}_{l+1}) - F(\mathcal{G}_l)$ is bounded by $F(X^*)$ for $X^* = \mathrm{argmax}_{X \in \mathcal{W}} F(X)$, i.e. the second candidate solution considered by the modified greedy algorithm. Hence

$$
F(\mathcal{G}_l) + F(X^*) \geq F(\mathcal{G}_{l+1}) \geq (1 - 1/e)F(OPT) - \beta\varepsilon
$$

and at least one of the values $F(X^*)$ or $F(\mathcal{G}_l)$ must be greater than or equal to $\frac{1}{2}((1 - 1/e)F(OPT) - \beta\varepsilon)$.

Using an argument analog to the one of [2], this analysis can be further tightened. Note that if there is a single (feasible, i.e., cost less than B) element X of value $F(X) \geq \frac{1}{2}F(OPT)$, Algorithm 1 will consider it, and produce a solution of value at least $\frac{1}{2}F(OPT) - 2\varepsilon$. Now assume for all X, $F(X) < \frac{1}{2}F(OPT)$. Let us consider two cases. In case $c(\mathcal{G}_l) < \frac{B}{2}$, then for all $X \notin \mathcal{G}_l$ it must hold that $c(X) > \frac{B}{2}$. Therefore, $|OPT \setminus \mathcal{G}_l| \leq 1$. Suppose $OPT \setminus \mathcal{G}_l = \{X'\}$. Since, by assumption, $F(X') < \frac{OPT}{2}$, by submodularity it must hold that $F(OPT \cap \mathcal{G}_l) > \frac{F(OPT)}{2}$. By monotonicity it follows that $F(\mathcal{G}_l) > \frac{F(OPT)}{2}$. For the last remaining case, namely $c(\mathcal{G}_l) \geq \frac{B}{2}$, Lemma 3 and the observation above for $\alpha = \frac{1}{2}$ imply that

$$
F(\mathcal{G}_l) \geq \left(1 - \prod_{k=1}^{l}\left(1 - \frac{c(X_k)}{B}\right)\right) F(OPT) - \beta\varepsilon \tag{3.1}
$$

$$
\geq \left(1 - \left(1 - \frac{1}{2l}\right)^{l}\right) F(OPT) - \beta\varepsilon \geq \left(1 - \frac{1}{\sqrt{e}}\right) F(OPT) - \beta\varepsilon \tag{3.2}
$$

$\square$

4 An improved approximation guarantee

To achieve the same performance guarantee of $(1 - 1/e)$ which can be achieved for the unit-cost in the case of general submodular functions [5], Khuller et.al. [2] propose a partial enumeration heuristic which enumerates all subsets of up to d elements for some constant $d > 0$, and complements these subsets using the modified greedy algorithm Algorithm 1. They prove that this algorithm guarantees a $(1 - 1/e)$ approximation for the budgeted MAX-k-COVER problem. The algorithm is stated below for general nondecreasing submodular functions:

```
    Input: $d > 0, B > 0, \mathcal{W} \subseteq \mathcal{V}$
    Output: Selection $\mathcal{A} \subseteq \mathcal{W}$
    begin
        $\mathcal{A}_1 := \text{argmax}\{F(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{W}, |\mathcal{A}| < d, c(\mathcal{A}) \leq B\} \; \mathcal{A}_2 := \emptyset;$
        foreach $\mathcal{G} \subseteq \mathcal{W}, |\mathcal{G}| = d, c(\mathcal{G}) \leq B$ do
            $\mathcal{W}' := \mathcal{W} \setminus \mathcal{G};$
            while $\mathcal{W}' \neq \emptyset$ do
                foreach $X \in \mathcal{W}'$ do $\Delta_X := \hat{F}'(\mathcal{G}; X);$
                $X^* := \text{argmax}\{\Delta_X / c(X) : X \in \mathcal{W}'\};$
                if $c(\mathcal{G}) + c(X^*) \leq B$ then $\mathcal{G} := \mathcal{G} \cup X^*;$
                $\mathcal{W}' := \mathcal{W}' \setminus X^*;$
            end
            if $F(\mathcal{G}) > F(\mathcal{A}_2)$ then $\mathcal{A}_2 := \mathcal{G}$
        end
        return $\underset{\mathcal{A} \in \{\mathcal{A}_1, \mathcal{A}_2\}}{\text{argmax}} \; F(\mathcal{A})$
    end
```

Algorithm 2: Approximation algorithm for budgeted case.

Theorem 4 (adapted from [2]). *Algorithm 2 achieves an approximation guarantee of*

$$(1 - 1/e)F(OPT) - \left(\frac{2B}{c_{min}}\right)\varepsilon$$

for (2.1) if sets at least up to cardinality $d = 3$ are enumerated, using $\mathcal{O}(|\mathcal{W}|^{d+2})$ evaluations of $\hat{F}'$.

Proof of Theorem 4. Assume that $|OPT| > d$, otherwise the algorithm finds the exact optimum. Reorder $OPT = \{Y_1, \ldots, Y_m\}$ such that

$$Y_{i+1} = \underset{Y \in OPT}{\text{argmax}} \, F(\{Y_1, \ldots, Y_i, Y\}) - F(\{Y_1, \ldots, Y_i\}),$$

and let $\mathcal{B} = \{Y_1, \ldots, Y_d\}$. Consider the iteration where the algorithm considers $\mathcal{B}$. Define the function

$$F'(\mathcal{A}) = F(\mathcal{A} \cup \mathcal{B}) - F(\mathcal{B}).$$

F' is a nondecreasing submodular set function with $F'(\emptyset) = 0$, hence we can apply the modified greedy algorithm to it. Let $\mathcal{A} = \{V_1, \ldots, V_l\}$ be the result of the algorithm, where V_i are chosen in sequence, let V_{l+1} be the first element from $OPT \setminus \mathcal{B}$ which could not be added due to budget constraints, and let $\mathcal{G} = \mathcal{A} \cup \mathcal{B}$. Per definition, $F(\mathcal{G}) = F'(\mathcal{A}) + F(\mathcal{B})$. Let $\Delta = F'(\mathcal{A} \cup V_{l+1}) - F'(\mathcal{A})$. Using Lemma 3, we find that

$$F'(\mathcal{A}) + \Delta \geq (1 - 1/e)F'(OPT \setminus \mathcal{B}) - \beta\varepsilon.$$

Furthermore observe, since the elements in OPT are ordered, that $F(\{Y_1, \ldots, Y_i\}) - F(\{Y_1, \ldots, Y_{i-1}\}) \geq \Delta$ for $1 \leq i \leq d$. Hence $F(\mathcal{B}) \geq d\Delta$. Now we get

$$
\begin{aligned}
F(\mathcal{G}) &= F(\mathcal{B}) + F'(\mathcal{A}) \\
&\geq F(\mathcal{B}) + (1 - 1/e)F'(OPT \setminus \mathcal{B}) - \Delta - \beta\varepsilon \\
&\geq F(\mathcal{B}) + (1 - 1/e)F'(OPT \setminus \mathcal{B}) - \frac{F(\mathcal{B})}{d} - \beta\varepsilon \\
&\geq (1 - 1/d)F(\mathcal{B}) + (1 - 1/e)F'(OPT \setminus \mathcal{B}) - \beta\varepsilon
\end{aligned}
$$

But by definition, $F(\mathcal{B}) + F'(OPT \setminus \mathcal{B}) = F(OPT)$, and hence for $d \geq 3$

$$F(\mathcal{G}) \geq (1 - 1/e)F(OPT) - \beta\varepsilon.$$

$\square$

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5 Conclusions

We presented an efficient approximation algorithm for the budgeted maximization of nondecreasing submodular set functions. We proved bounds on the absolute error which are incurred if the marginal increases can only be computed with an absolute error. We believe that our results are useful for the wide class of combinatorial optimization problems concerned with maximizing submodular functions.

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