Efficient Sampling for Learning Sparse Additive Models in High Dimensions

Hemant Tyagi
ETH Zürich
htyagi@inf.ethz.ch

Andreas Krause
ETH Zürich
krausea@ethz.ch

Bernd Gärtner
ETH Zürich
gaertner@inf.ethz.ch

Abstract

We consider the problem of learning sparse additive models, i.e., functions of the form: $f(\mathbf{x}) = \sum_{l \in S} \phi_l(x_l)$, $\mathbf{x} \in \mathbb{R}^d$ from point queries of f. Here S is an unknown subset of coordinate variables with $|S| = k \ll d$. Assuming ϕ_l's to be smooth, we propose a set of points at which to sample f and an efficient randomized algorithm that recovers a *uniform approximation* to each unknown ϕ_l. We provide a rigorous theoretical analysis of our scheme along with sample complexity bounds. Our algorithm utilizes recent results from compressive sensing theory along with a novel convex quadratic program for recovering robust uniform approximations to univariate functions, from point queries corrupted with arbitrary bounded noise. Lastly we theoretically analyze the impact of noise – either arbitrary but bounded, or stochastic – on the performance of our algorithm.

1 Introduction

Several problems in science and engineering require estimating a real-valued, non-linear (and often non-convex) function f defined on a compact subset of $\mathbb{R}^d$ in high dimensions. This challenge arises, e.g., when characterizing complex engineered or natural (e.g., biological) systems [1, 2, 3]. The numerical solution of such problems involves learning the unknown f from point evaluations $(\mathbf{x}_i, f(x_i))_{i=1}^n$. Unfortunately, if the only assumption on f is of mere smoothness, then the problem is in general intractable. For instance, it is well known [4] that if f is C^s-smooth then $n = \Omega((1/\delta)^{d/s})$ samples are needed for uniformly approximating f within error $0 < \delta < 1$. This exponential dependence on d is referred to as the curse of dimensionality.

Fortunately, many functions arising in practice are much better behaved in the sense that they are intrinsically low-dimensional, i.e., depend on only a small subset of the d variables. Estimating such functions has received much attention and has led to a considerable amount of theory along with algorithms that do not suffer from the curse of dimensionality (cf., [5, 6, 7, 8]). Here we focus on the problem of learning one such class of functions, assuming f possesses the *sparse* additive structure:

$$f(x_1, x_2, \ldots, x_d) = \sum_{l \in S} \phi_l(x_l); \quad S \subset \{1, \ldots, d\}, |S| = k \ll d. \tag{1.1}$$

Functions of the form (1.1) are referred to as sparse additive models (SPAMs) and generalize sparse linear models to which they reduce to if each ϕ_l is linear. The problem of estimating SPAMs has received considerable attention in the regression setting (cf., [9, 10, 11] and references within) where $(x_i, f(x_i))_{i=1}^n$ are typically i.i.d samples from some unknown probability measure $\mathbb{P}$. This setting, however, does not consider the possibility of sampling f at specifically chosen points, tailored to the additive structure of f. In this paper, we propose a strategy for querying f, together with an efficient recovery algorithm, with much stronger guarantees than known in the regression setting. In particular, we provide the first results guaranteeing *uniformly accurate recovery of each individual component* ϕ_l of the SPAM. This can be crucial in applications where the goal is to not merely approximate f, but gain insight into its structure.

Related work. SPAMs have been studied extensively in the regression setting, with observations being corrupted with random noise. [9] proposed the COSSO method, which is an extension of the Lasso to the reproducing kernel Hilbert space (RKHS) setting. A similiar extension was considered in [10]. In [12], the authors propose a least squares method regularized with smoothness, with each ϕ_l lying in an RKHS and derive error rates for estimating f, in the $L^2(\mathbb{P})$ norm[1]. [13, 14] propose methods based on least squares loss regularized with sparsity and smoothness constraints. [13] proves consistency of its method in terms of mean squared risk while [14] derives error rates for estimating f in the empirical $L^2(\mathbb{P}_n)$ norm [1]. [11] considers the setting where each ϕ_l lies in an RKHS. They propose a convex program for estimating f and derive error rates for the same, in the $L^2(\mathbb{P}), L^2(\mathbb{P}_n)$ norms. Furthermore they establish the minimax optimality of their method for the $L^2(\mathbb{P})$ norm. For instance, they derive an error rate of $O((k \log d/n) + kn^{-\frac{2s}{2s+1}})$ in the $L^2(\mathbb{P})$ norm for estimating C^s smooth SPAMs. An estimator similar to the one in [11] was also considered by [15]. They derive similar error rates as in [11], albeit under stronger assumptions on f. [16] proposes a method based on the adaptive group Lasso which asymptotically recovers S as n increases. They also derive $L^2(\mathbb{P})$ error rates for the individual components of the SPAM.

There is further related work in approximation theory, where it is assumed that f can be sampled at a desired set of points. [5] considers a setting more general than (1.1), with f simply assumed to depend on an unknown subset of $k \ll d$-coordinate variables. They construct a set of sampling points of size $O(c^k \log d)$ for some constant $c > 0$, and present an algorithm that recovers a uniform approximation[2] to f. This model is generalized in [8], with f assumed to be of the form $f(x) = g(\mathbf{A}\mathbf{x})$ for unknown $\mathbf{A} \in \mathbb{R}^{k \times d}$; each row of $\mathbf{A}$ is assumed to be sparse. [7] generalizes this, by removing the sparsity assumption on $\mathbf{A}$. While the methods of [5, 8, 7] could be employed for learning SPAMs, their sampling sets will be of size exponential in k, and hence sub-optimal. Furthermore, while these methods derive uniform approximations to f, they are unable to recover the individual ϕ_l's.

Our contributions. Our contributions are threefold:

1. We propose an efficient algorithm that queries f at $O(k \log d)$ locations and recovers: (i) the active set S exactly, along with (ii) a *uniform approximation* to each ϕ_l, $l \in S$. In contrast, the existing error bounds in the statistics community [11, 12, 15] are in the much weaker $L^2(\mathbb{P})$ sense. Furthermore, the existing results in approximation theory provide explicit error bounds for recovering f and not the individual ϕ_l's.

2. An important component of our algorithm is a novel convex quadratic program for estimating an unknown univariate function from point queries corrupted with *arbitrary* bounded noise. We derive rigorous error bounds for this program in the L^∞ norm that demonstrate the robustness of the solution returned. We also explicitly demonstrate the effect of noise, sampling density and the curvature of the function on the solution returned.

3. We theoretically analyze the impact of additive noise in the point queries on the performance of our algorithm, for two noise models: arbitrary bounded noise and stochastic (iid) noise. In particular for additive Gaussian noise, we show that our algorithm recovers a robust *uniform approximation* to each ϕ_l with at most $O(k^3 (\log d)^2)$ point queries of f. We also provide simulation results that validate our theoretical findings.

2 Problem statement

For any function g we denote its pth derivative by $g^{(p)}$ when p is large, else we use appropriate number of prime symbols. $\| g \|_{L^\infty[a,b]}$ denotes the L^∞ norm of g in $[a,b]$. For a vector $\mathbf{x}$ we denote its ℓ_q norm for $1 \leq q \leq \infty$ by $\| \mathbf{x} \|_q$. We consider approximating functions $f : \mathbb{R}^d \to \mathbb{R}$ from point queries. In particular, for some unknown active $S \subset \{1, \ldots, d\}$ with $|S| = k \ll d$, we assume f to be of the additive form: $f(x_1, \ldots, x_d) = \sum_{l \in S} \phi_l(x_l)$. Here $\phi_l : \mathbb{R} \to \mathbb{R}$ are the individual univariate components of the model. Our goal is to query f at suitably chosen points in its domain in order to recover an estimate $\phi_{\text{est},l}$ of ϕ_l in a compact subset $\Omega \subset \mathbb{R}$ for each $l \in S$. We measure the approximation error in the L^∞ norm. For simplicity, we assume that $\Omega = [-1, 1]$,

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[1] $\| f \|_{L^2(\mathbb{P})}^2 = \int |f(\mathbf{x})|^2 \, d\mathbb{P}(\mathbf{x})$ and $\| f \|_{L^2(\mathbb{P}_n)}^2 = \frac{1}{n} \sum_i f^2(\mathbf{x}_i)$

[2] This means in the L^∞ norm

meaning that we guarantee an upper bound on: $\| \phi_{\text{est},l} - \phi_l \|_{L^\infty[-1,1]}$; $\quad l \in S$. Furthermore, we assume that we can query f from a slight enlargement: $[-(1+r),(1+r)]^d$ of $[-1,1]^d$ for[3] some small $r > 0$. As will be seen later, the enlargement r can be made arbitrarily close to 0. We now list our main assumptions for this problem.

1. Each ϕ_l is assumed to be sufficiently smooth. In particular we assume that $\phi_l \in C^5[-(1+r),(1+r)]$ where C^5 denotes five times continuous differentiability. Since $[-(1+r),(1+r)]$ is compact, this implies that there exist constants $B_1, \ldots, B_5 \geq 0$ so that

$$\max_{l \in S} \| \phi_l^{(p)} \|_{L^\infty[-(1+r),(1+r)]} \leq B_p; \quad p = 1, \ldots, 5. \tag{2.1}$$

2. We assume each ϕ_l to be centered in the interval $[-1,1]$, i.e. $\int_{-1}^1 \phi_l(t)dt = 0$; $l \in S$. Such a condition is necessary for unique identification of ϕ_l. Otherwise one could simply replace each ϕ_l with $\phi_l + a_l$ for $a_l \in \mathbb{R}$ where $\sum_l a_l = 0$ and unique identification will not be possible.

3. We require that for each ϕ_l, $\exists I_l \subseteq [-1,1]$ with I_l being connected and $\mu(I_l) \geq \delta$ so that $|\phi_l'(x)| \geq D$; $\forall x \in I_l$. Here $\mu(I)$ denotes the Lebesgue measure of I and $\delta, D > 0$ are constants assumed to be known to the algorithm. This assumption essentially enables us to detect the active set S. If say ϕ_l' was zero or close to zero throughout $[-1,1]$ for some $l \in S$, then due to Assumption 2 this would imply that ϕ_l is zero or close to zero.

We remark that it suffices to use estimates for our problem parameters instead of exact values. In particular we can use upper bounds for: k, B_p; $p = 1, \ldots, 5$ and lower bounds for the parameters: D, δ. Our methods and results stated in the coming sections will remain unchanged.

3   Our sampling scheme and algorithm

In this section, we first motivate and describe our sampling scheme for querying f. We then outline our algorithm and explain the intuition behind its different stages. Consider the Taylor expansion of f at any point $\xi \in \mathbb{R}^d$ along the direction $\mathbf{v} \in \mathbb{R}^d$ with *step size*: $\epsilon > 0$. For any C^p smooth f; $p \geq 2$, we obtain for $\zeta = \xi + \theta\mathbf{v}$ for some $0 < \theta < \epsilon$ the following expression:

$$\frac{f(\xi + \epsilon\mathbf{v}) - f(\xi)}{\epsilon} = \langle \mathbf{v}, \nabla f(\xi) \rangle + \frac{1}{2}\epsilon\mathbf{v}^T \nabla^2 f(\zeta)\mathbf{v}. \tag{3.1}$$

Note that (3.1) can be interpreted as taking a *noisy linear* measurement of $\nabla f(\xi)$ with the measurement vector $\mathbf{v}$ and the noise being the Taylor remainder term. Importantly, due to the sparse additive form of f, we have $\phi_l \equiv 0, l \notin S$, implying that $\nabla f(\xi) = [\phi_1'(\xi_1) \ \phi_2'(\xi_2) \ldots \phi_d'(\xi_d)]$ is at most k-sparse. Hence (3.1) actually represents a noisy linear measurement of the k-sparse vector : $\nabla f(\xi)$. For any fixed ξ, we know from compressive sensing (CS) [17, 18] that $\nabla f(\xi)$ can be recovered (with high probability) using few random linear measurements[4].

This motivates the following sets of points using which we query f as illustrated in Figure 1. For integers $m_x, m_v > 0$ we define

$$\mathcal{X} := \left\{ \xi_i = \frac{i}{m_x}(1,1,\ldots,1)^T \in \mathbb{R}^d : i = -m_x, \ldots, m_x \right\}, \tag{3.2}$$

$$\mathcal{V} := \left\{ v_j \in \mathbb{R}^d : v_{j,l} = \pm\frac{1}{\sqrt{m_v}} \text{ w.p. } 1/2 \text{ each}; \ j = 1, \ldots, m_v \text{ and } l = 1, \ldots, d \right\}. \tag{3.3}$$

Using (3.1) at each $\xi_i \in \mathcal{X}$ and $\mathbf{v}_j \in \mathcal{V}$ for $i = -m_x, \ldots, m_x$ and $j = 1, \ldots, m_v$ leads to:

$$\underbrace{\frac{f(\xi_i + \epsilon\mathbf{v}_j) - f(\xi_i)}{\epsilon}}_{y_{i,j}} = \langle \mathbf{v}_j, \underbrace{\nabla f(\xi_i)}_{\mathbf{x}_i} \rangle + \underbrace{\frac{1}{2}\epsilon\mathbf{v}_j^T \nabla^2 f(\zeta_{i,j})\mathbf{v}_j}_{n_{i,j}}, \tag{3.4}$$

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[3] In case $f : [a,b]^d \to \mathbb{R}$ we can define $g : [-1,1]^d \to \mathbb{R}$ where $g(\mathbf{x}) = f(\frac{(b-a)}{2}\mathbf{x} + \frac{b+a}{2}) = \sum_{l \in S} \tilde{\phi}_l(x_l)$ with $\tilde{\phi}_l(x_l) = \phi_l(\frac{(b-a)}{2}x_l + \frac{b+a}{2})$. We then sample g from within $[-(1+r),(1+r)]^d$ for some small $r > 0$ by querying f, and estimate $\tilde{\phi}_l$ in $[-1,1]$ which in turn gives an estimate to ϕ_l in $[a,b]$.

[4] Estimating sparse gradients via compressive sensing has been considered previously by Fornasier et al. [8] albeit for a substantially different function class than us. Hence their sampling scheme differs considerably from ours, and is not tailored for learning SPAMs.

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where $\mathbf{x}_i = \bigtriangledown f(\xi_i) = [\phi_1'(i/m_x) \ \phi_2'(i/m_x) \dots \phi_d'(i/m_x)]$ is k-sparse. Let us denote $\mathbf{V} = [\mathbf{v}_1 \dots \mathbf{v}_{m_v}]^T$, $\mathbf{y}_i = [y_{i,1} \dots y_{i,m_v}]$ and $\mathbf{n}_i = [n_{i,1} \dots n_{i,m_v}]$. Then for each i, we can write (3.4) in the succinct form:

$$\mathbf{y}_i = \mathbf{V}\mathbf{x}_i + \mathbf{n}_i. \tag{3.5}$$

Here $\mathbf{V} \in \mathbb{R}^{m_v \times d}$ represents the linear measurement matrix, $\mathbf{y}_i \in \mathbb{R}^{m_v}$ denotes the measurement vector at ξ_i and $\mathbf{n}_i$ represents "noise" on account of non-linearity of f. Note that we query f at $|\mathcal{X}|(|\mathcal{V}| + 1) = (2m_x + 1)(m_v + 1)$ many points. Given $\mathbf{y}_i, \mathbf{V}$ we can recover a robust approximation to $\mathbf{x}_i$ via ℓ_1 minimization [17, 18]. On account of the structure of $\bigtriangledown f$, we thus recover noisy estimates to ϕ_l' at equispaced points along the interval $[-1, 1]$. We are now in a position to formally present our algorithm for learning SPAMs.

Our algorithm for learning SPAMs. The steps involved in our learning scheme are outlined in Algorithm 1. Steps 1-4 involve the CS-based recovery stage wherein we use the aforementioned sampling sets to formulate our problem as a CS one. Step 4 involves a simple thresholding procedure where an appropriate threshold τ is employed to recover the unknown active set S. In Section 4 we provide precise conditions on our sampling parameters which guarantee exact recovery, i.e. $\widehat{S} = S$. Step 5 leverages a convex quadratic program (P), that uses noisy estimates of $\phi_l'(i/m_x)$, i.e., $\widehat{x}_{i,l}$ for each $l \in \widehat{S}$ and $i = -m_x, \dots, m_x$, to return a cubic spline estimate $\tilde{\phi}_l'$. This program and its theoretical properties are explained in Section 4. Finally, in Step 6 we derive our final estimate $\phi_{\text{est},l}$ via piecewise integration of $\tilde{\phi}_l'$ for each $l \in \widehat{S}$. Hence our final estimate of ϕ_l is a spline of degree 4. The performance of Algorithm 1 for recovering S and the individual ϕ_l's is presented in Theorem 1, which is also our first main result. All proofs are deferred to the appendix.
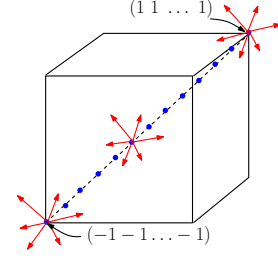

Figure 1: The points $\xi_i \in \mathcal{X}$ (blue disks) and $\xi_i + \epsilon\mathbf{v}_j$ (red arrows) for $\mathbf{v}_j \in \mathcal{V}$.

Algorithm 1 Algorithm for learning ϕ_l in the SPAM: $f(\mathbf{x}) = \sum_{l \in S} \phi_l(x_l)$

1: Choose m_x, m_v and construct sampling sets $\mathcal{X}$ and $\mathcal{V}$ as in (3.2), (3.3).
2: Choose step size $\epsilon > 0$. Query f at $f(\xi_i), f(\xi_i + \epsilon\mathbf{v}_j)$ for $i = -m_x, \dots, m_x$ and $j = 1, \dots, m_v$.
3: Construct $\mathbf{y}_i$ where $y_{i,j} = \frac{f(\xi_i + \epsilon\mathbf{v}_j) - f(\xi_i)}{\epsilon}$ for $i = -m_x, \dots, m_x$ and $j = 1, \dots, m_v$.
4: Set $\widehat{\mathbf{x}}_i := \underset{\mathbf{y}_i = \mathbf{V}\mathbf{z}}{\operatorname{argmin}} \| \mathbf{z} \|_1$. For $\tau > 0$ compute $\widehat{S} = \cup_{i=-m_x}^{m_x} \{l \in \{1, \dots, d\} : |\widehat{x}_{i,l}| > \tau\}$.
5: For each $l \in \widehat{S}$, run (P) as defined in Section 4 using $(\widehat{x}_{i,l})_{i=-m_x}^{m_x}$, τ and some smoothing parameter $\gamma \geq 0$, to obtain $\tilde{\phi}_l'$.
6: For each $l \in \widehat{S}$, set $\phi_{\text{est},l}$ to be the piece-wise integral of $\tilde{\phi}_l'$ as explained in Section 4.

Theorem 1. *There exist constants $C, C_1 > 0$ such that if $m_x \geq (1/\delta)$, $m_v \geq C_1 k \log d$, $0 < \epsilon < \frac{D\sqrt{m_v}}{CkB_2}$ and $\tau = \frac{C\epsilon k B_2}{2\sqrt{m_v}}$ then with high probability, $\widehat{S} = S$ and for any $\gamma \geq 0$ the estimate $\phi_{\text{est},l}$ returned by Algorithm 1 satisfies for each $l \in S$:*

$$\| \phi_{\text{est},l} - \phi_l \|_{L^\infty[-1,1]} \leq [59(1 + \gamma)] \frac{C\epsilon k B_2}{\sqrt{m_v}} + \frac{87}{64 m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]}. \tag{3.6}$$

Recall that k, B_2, D, δ are our problem parameters introduced in Section 2, while ϵ is the step size parameter from (3.4). We see that with $O(k \log d)$ point queries of f and with $\epsilon < \frac{D\sqrt{m_v}}{CkB_2}$, the active set is recovered exactly. The error bound in (3.6) holds for all such choices of ϵ. It is a sum of two terms in which the first one arises during the estimation of $\bigtriangledown f$ during the CS stage. The second error term is the interpolation error bound for interpolating ϕ_l' from its samples in the noise-free setting. We note that our point queries lie in $[-(1 + (\epsilon/\sqrt{m_v})), (1 + (\epsilon/\sqrt{m_v}))]^d$. For the stated condition on ϵ in Theorem 1 we have $\epsilon/\sqrt{m_v} < \frac{D}{CkB_2}$ which can be made arbitrarily close to zero by choosing an appropriately small ϵ. Hence we sample from only a small enlargement of $[-1, 1]^d$.

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4 Analyzing the algorithm

We now describe and analyze in more detail the individual stages of Algorithm 1. We first analyze Steps 1-4 which constitute the compressive sensing (CS) based recovery stage. Next, we analyze Step 5 where we also introduce our convex quadratic program. Lastly, we analyze Step 6 where we derive our final estimate $\phi_{\text{est},l}$.

Compressive sensing-based recovery stage. This stage of Algorithm 1 involves solving a sequence of linear programs for recovering estimates of $\mathbf{x}_i = [\phi'_1(i/m_x) \ \ldots \ \phi'_d(i/m_x)]$ for $i = -m_x, \ldots, m_x$. We note that the measurements $\mathbf{y}_i$ are noisy linear measurements of $\mathbf{x}_i$ with the noise being arbitrary and bounded. For such a noise model, it is known that ℓ_1 minimization results in robust recovery of the sparse signal [19]. Using this result in our setting allows us to quantify the recovery error $\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2$ as specified in Lemma 1.

Lemma 1. *There exist constants $c'_3 \geq 1$ and $C, c'_1 > 0$ such that for m_v satisfying $c'_3 k \log d < m_v < d/(\log 6)^2$ we have with probability at least $1 - e^{-c'_1 m_v} - e^{-\sqrt{m_v d}}$ that $\widehat{\mathbf{x}}_i$ satisfies $\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2 \leq \frac{C \epsilon k B_2}{2\sqrt{m_v}}$ for all $i = -m_x, \ldots, m_x$. Furthermore, given that this holds and $m_x \geq 1/\delta$ is satisfied we then have for any $\epsilon < \frac{D\sqrt{m_v}}{CkB_2}$ that the choice $\tau = \frac{C \epsilon k B_2}{2\sqrt{m_v}}$ implies that $\widehat{S} = S$.*

Thus upon using ℓ_1 minimization based decoding at $2m_x + 1$ points, we recover robust estimates $\widehat{\mathbf{x}}_i$ to $\mathbf{x}_i$ which immediately gives us estimates $\widehat{\phi}'_l(i/m_x) = \widehat{\mathbf{x}}_{i,l}$ of $\phi'_l(i/m_x)$ for $i = -m_x, \ldots, m_x$ and $l = 1, \ldots, d$. In order to recover the active set S, we first note that the spacing between consecutive samples in $\mathcal{X}$ is $1/m_x$. Therefore the condition $m_x \geq 1/\delta$ implies on account of Assumption 3 that the sample spacing is fine enough to ensure that for each $l \in S$, there exists a sample i for which $|\phi'_l(i/m_x)| \geq D$ holds. The stated choice of the step size ϵ essentially guarantees $\forall l \notin S, i$ that $\left|\widehat{\phi}'_l(i/m_x)\right|$ lies within a sufficiently small neighborhood of the origin in turn enabling detection of the active set. Therefore after this stage of Algorithm 1, we have at hand: the active set S along with the estimates: $(\widehat{\phi}'_l(i/m_x))_{i=m_x}^{m_x}$ for each $l \in S$. Furthermore, it is easy to see that $\left|\widehat{\phi}'_l(i/m_x) - \phi'_l(i/m_x)\right| \leq \tau = \frac{C \epsilon k B_2}{2\sqrt{m_v}}, \forall l \in S, \forall i$.

Robust estimation via cubic splines. Our aim now is to recover a smooth, robust estimate to ϕ'_l by using the noisy samples $(\widehat{\phi}'_l(i/m_x))_{i=m_x}^{m_x}$. Note that the noise here is arbitrary and bounded by $\tau = \frac{C \epsilon k B_2}{2\sqrt{m_v}}$. To this end we choose to use cubic splines as our estimates, which are essentially piecewise cubic polynomials that are C^2 smooth [20]. There is a considerable amount of literature in the statistics community devoted to the problem of estimating univariate functions from noisy samples via cubic splines (cf., [21, 22, 23, 24]), albeit under the setting of random noise. Cubic splines have also been studied extensively in the approximation theoretic setting for *interpolating* samples (cf., [20, 25, 26]).

We introduce our solution to this problem in a more general setting. Consider a smooth function $g : [t_1, t_2] \to \mathbb{R}$ and a uniform mesh[5]: $\prod : t_1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = t_2$ with $x_i - x_{i-1} = h$. We have at hand noisy samples: $\widehat{g}_i = g(x_i) + e_i$, with noise e_i being arbitrary and bounded: $|e_i| \leq \tau$. In the noiseless scenario, the problem would be an interpolation one for which a popular class of cubic splines are the "not-a-knot" cubic splines [25]. These achieve optimal $O(h^4)$ error rates for C^4 smooth g without using any higher order information about g as boundary conditions. Let $H^2[t_1, t_2]$ denote the space of cubic splines defined on $[t_1, t_2]$ w.r.t $\prod$. We then propose finding the cubic spline estimate as a solution of the following convex optimization problem (in the $4n$ coefficients of the n cubic polynomials) for some parameter $\gamma \geq 0$:

$$(P) \begin{cases} \min_{\mathcal{L} \in H^2[t_1,t_2]} \int_{t_1}^{t_2} \mathcal{L}''(x)^2 dx & (4.1) \\ \text{s.t.} \quad \widehat{g}_i - \gamma\tau \leq \mathcal{L}(x_i) \leq \widehat{g}_i + \gamma\tau; \quad i = 0, \ldots, n, & (4.2) \\ \mathcal{L}'''(x_1^-) = \mathcal{L}'''(x_1^+), \quad \mathcal{L}'''(x_{n-1}^-) = \mathcal{L}'''(x_{n-1}^+). & (4.3) \end{cases}$$

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[5]We consider uniform meshes for clarity of exposition. The results in this section can be easily generalized to non-uniform meshes.

Note that (P) is a convex QP with linear constraints. The objective function can be verified to be a positive definite quadratic form in the spline coefficients[6]. Specifically, the objective measures the total curvature of a feasible cubic spline in $[t_1, t_2]$. Each of the constraints (4.2)-(4.3) along with the implicit continuity constraints of $\mathcal{L}^{(p)}$; $p = 0, 1, 2$ at the interior points of $\prod$, are linear equalities/inequalities in the coefficients of the piecewise cubic polynomials. (4.3) refers to the not-a-knot boundary conditions [25] which are also linear equalities in the spline coefficients. These conditions imply that $\mathcal{L}'''$ is continuous[7] at the knots x_1, x_{n-1}. Thus, (P) searches amongst the space of all not-a-knot cubic splines such that $\mathcal{L}(x_i)$ lies within a $\pm\gamma\tau$ interval of $\widehat{g}_i$, and returns the smoothest solution, i.e., the one with the least total curvature. The parameter $\gamma \geq 0$, controls the degree of smoothness of the solution. Clearly, $\gamma = 0$ implies interpolating the noisy samples $(\widehat{g}_i)_{i=0}^n$. As γ increases, the search interval: $[\widehat{g}_i - \gamma\tau, \widehat{g}_i + \gamma\tau]$ becomes larger for all i, leading to smoother feasible cubic splines. The following theorem formally describes the estimation properties of (P) and is also our second main result.

Theorem 2. *For $g \in C^4[t_1, t_2]$ let $\mathcal{L}^* : [t_1, t_2] \to \mathbb{R}$ be a solution of (P) for some parameter $\gamma \geq 0$. We then have that*

$$\| \mathcal{L}^* - g \|_\infty \leq \left[\frac{118(1 + \gamma)}{3} \right] \tau + \frac{29}{64} h^4 \| g^{(4)} \|_\infty . \tag{4.4}$$

We show in the appendix that if $\int_{t_1}^{t_2} (\mathcal{L}^{*''}(x))^2 dx > 0$, then $\mathcal{L}^*$ is unique. Note that the error bound (4.4) is a sum of two terms. The first term is proportional to the external noise bound: τ, indicating that the solution is *robust* to noise. The second term is the error that would arise even if perturbation was absent i.e. $\tau = 0$. Intuitively, if $\gamma\tau$ is large enough, then we would expect the solution returned by (P) to be a line. Indeed, a larger value of $\gamma\tau$ would imply a larger search interval in (4.2), which if sufficiently large, would allow a line (that has zero curvature) to lie in the feasible region. More formally, we show in the appendix, sufficient conditions: $\tau = \Omega(\frac{n^{1/2}\|g''\|_\infty}{\gamma - 1})$, $\gamma > 1$, which if satisfied, imply that the solution returned by (P) is a line. This indicates that if either n is small or g has small curvature, then moderately large values of τ and/or γ will cause the solution returned by (P) to be a line. If an estimate of $\| g'' \|_\infty$ is available, then one could for instance, use the upper bound $1 + O(n^{1/2} \| g'' \|_\infty /\tau)$ to restrict the range of values of γ within which (P) is used.

Theorem 2 has the following Corollary for estimation of C^4 smooth ϕ_l' in the interval $[-1, 1]$. The proof simply involves replacing: g with ϕ_l', $n + 1$ with $2m_x + 1$, h with $1/m_x$ and τ with $\frac{C\epsilon kB_2}{2\sqrt{m_v}}$. As the perturbation τ is directly proportional to the step size ϵ, we show in the appendix that if additionally $\epsilon = \Omega(\frac{\sqrt{m_x m_v}\|\phi_l''\|_\infty}{\gamma - 1})$, $\gamma > 1$, holds, then the corresponding estimate $\tilde{\phi}_l'$ will be a line.

Corollary 1. *Let (P) be employed for each $l \in S$ using noisy samples $\left\{ \widehat{\phi}_l'(i/m_x) \right\}_{i=-m_x}^{m_x}$, and with step size ϵ satisfying $0 < \epsilon < \frac{D\sqrt{m_v}}{CkB_2}$. Denoting $\tilde{\phi}_l'$ as the corresponding solution returned by (P), we then have for any $\gamma \geq 0$ that:*

$$\| \tilde{\phi}_l' - \phi_l' \|_{L^\infty[-1,1]} \leq \left[\frac{59(1 + \gamma)}{3} \right] \frac{C\epsilon kB_2}{\sqrt{m_v}} + \frac{29}{64 m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} . \tag{4.5}$$

The final estimate. We now derive the final estimate $\phi_{\text{est},l}$ of ϕ_l for each $l \in S$. Denote $x_0(= -1) < x_1 < \cdots < x_{2m_x-1} < x_{2m_x}(= 1)$ as our equispaced set of points on $[-1, 1]$. Since $\tilde{\phi}_l' : [-1, 1] \to \mathbb{R}$ returned by (P) is a cubic spline, we have $\tilde{\phi}_l'(x) = \tilde{\phi}_{l,i}'(x)$ for $x \in [x_i, x_{i+1}]$ where $\tilde{\phi}_{l,i}'$ is a polynomial of degree at most 3. We then define $\phi_{\text{est},l}(x) := \tilde{\phi}_{l,i}(x) + F_i$ for $x \in [x_i, x_{i+1}]$ and $i = 0, \ldots, 2m_x - 1$. Here $\tilde{\phi}_{l,i}$ is an antiderivative of $\tilde{\phi}_{l,i}'$ and F_i's are constants of integration. Denoting $F_0 = F$, we have that $\phi_{\text{est},l}$ is continuous at $x_1, \ldots, x_{2m_x-1}$ for: $F_i = \tilde{\phi}_{l,0}(x_1) + \sum_{j=1}^{i-1}(\tilde{\phi}_{l,j}(x_{j+1}) - \tilde{\phi}_{l,j}(x_j)) - \tilde{\phi}_{l,i}(x_i) + F = F_i' + F$; $1 \leq i \leq 2m_x - 1$. Hence by denoting $\psi_{l,i}(\cdot) := \tilde{\phi}_{l,i}(\cdot) + F_i'$ we obtain $\phi_{\text{est},l}(\cdot) = \psi_l(\cdot) + F$ where $\psi_l(x) = \psi_{l,i}(x)$ for

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[6]Shown in the appendix.

[7]$f(x^-) = \lim_{h \to 0^-} f(x + h)$ and $f(x^+) = \lim_{h \to 0^+} f(x + h)$ denote left,right hand limits respectively.

$x \in [x_i, x_{i+1}]$. Now on account of Assumption 2, we require $\phi_{\text{est},l}$ to also be centered implying $F = -\frac{1}{2} \int_{-1}^{1} \psi_l(x) dx$. Hence we output our final estimate of ϕ_l to be:

$$\phi_{\text{est},l}(x) := \psi_l(x) - \frac{1}{2} \int_{-1}^{1} \psi_l(x) dx; \quad x \in [-1, 1]. \tag{4.6}$$

Since $\phi_{\text{est},l}$ is by construction continuous in $[-1, 1]$, is a piecewise combination of polynomials of degree at most 4, and since $\phi'_{\text{est},l}$ is a cubic spline, $\phi_{\text{est},l}$ is a spline function of order 4. Lastly, we show in the proof of Theorem 1 that $\| \phi_{\text{est},l} - \phi_l \|_{L^\infty[-1,1]} \leq 3 \| \tilde{\phi}'_l - \phi'_l \|_{L^\infty[-1,1]}$ holds. Using Corollary 1, this provides us with the error bounds stated in Theorem 1.

5 Impact of noise on performance of our algorithm

Our third main contribution involves analyzing the more realistic scenario, when the point queries are corrupted with additive external noise z'. Thus querying f in Step 2 of Algorithm 1 results in noisy values: $f(\xi_i) + z'_i$ and $f(\xi_i + \epsilon \mathbf{v}_j) + z'_{i,j}$ respectively. This changes (3.5) to the noisy linear system: $\mathbf{y}_i = \mathbf{V}\mathbf{x}_i + \mathbf{n}_i + \mathbf{z}_i$ where $z_{i,j} = (z'_{i,j} - z'_i)/\epsilon$ for $i = -m_x, \ldots, m_x$ and $j = 1, \ldots, m_v$. Notice that external noise gets scaled by $(1/\epsilon)$, while $|n_{i,j}|$ scales linearly with ϵ.

Arbitrary bounded noise. In this model, the external noise is arbitrary but bounded, so that $|z'_i|, |z'_{i,j}| < \kappa; \forall i, j$. It can be verified along the lines of the proof of Lemma 1 that: $\| \mathbf{n}_i + \mathbf{z}_i \|_2 \leq \sqrt{m_v} \left(\frac{2\kappa}{\epsilon} + \frac{\epsilon k B_2}{2m_v} \right)$. Observe that unlike the noiseless setting, ϵ cannot be made arbitrarily close to 0, as it would blow up the impact of the external noise. The following theorem shows that if κ is small relative to $D^2 < |\phi'_l(x)|^2, \forall x \in I_l, l \in S$, then[8] there exists an interval for choosing ϵ, within which Algorithm 1 recovers exactly the active set S. This condition has the natural interpretation that if the signal-to-'external noise' ratio in I_l is sufficiently large, then S can be detected exactly.

Theorem 3. *There exist constants $C, C_1 > 0$ such that if $\kappa < D^2/(16C^2 k B_2)$, $m_x \geq (1/\delta)$, and $m_v \geq C_1 k \log d$ hold, then for any $\epsilon \in \frac{D\sqrt{m_v}}{2CkB_2}[1 - A, 1 + A]$ where $A := \sqrt{1 - (16C^2 k B_2 \kappa)/D^2}$ and $\tau = \sqrt{m_v} \left(\frac{2\kappa}{\epsilon} + \frac{\epsilon k B_2}{2m_v} \right)$, we have in Algorithm 1, with high probability, that $\widehat{S} = S$ and for any $\gamma \geq 0$, for each $l \in S$:*

$$\| \phi_{\text{est},l} - \phi_l \|_{L^\infty[-1,1]} \leq [59(1 + \gamma)] \left(\frac{4C\sqrt{m_v}\kappa}{\epsilon} + \frac{C\epsilon k B_2}{\sqrt{m_v}} \right) + \frac{87}{64m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]}. \tag{5.1}$$

Stochastic noise. In this model, the external noise is assumed to be i.i.d. Gaussian, so that $z'_i, z'_{i,j} \sim \mathcal{N}(0, \sigma^2)$; i.i.d. $\forall i, j$. In this setting we consider resampling f at the query point N times and then averaging the noisy samples, in order to reduce σ. Given this, we now have that $z'_i, z'_{i,j} \sim \mathcal{N}(0, \frac{\sigma^2}{N})$; i.i.d. $\forall i, j$. Using standard tail-bounds for Gaussians, we can show that for any $\kappa > 0$ if N is chosen large enough then: $|z_{i,j}| = |z'_i - z'_{i,j}| \leq 2\kappa; \forall i, j$ with high probability. Hence the external noise $z_{i,j}$ would be bounded with high probability and the analysis for Theorem 3 can be used in a straightforward manner. Of course, an advantage that we have in this setting is that κ can be chosen to be arbitrarily close to zero by choosing a correspondingly large value of N. We state all this formally in the form of the following theorem.

Theorem 4. *There exist constants $C, C_1 > 0$ such that for $\kappa < D^2/(16C^2 k B_2)$, $m_x \geq (1/\delta)$, and $m_v \geq C_1 k \log d$, if we re-sample each query in Step 2 of Algorithm 1: $N > \frac{\sigma^2}{\kappa^2} \log \left(\frac{\sqrt{2}\sigma}{\kappa p} |\mathcal{X}| |\mathcal{V}| \right)$ times for $0 < p < 1$, and average the values, then for any $\epsilon \in \frac{D\sqrt{m_v}}{2CkB_2}[1 - A, 1 + A]$ where $A := \sqrt{1 - (16C^2 k B_2 \kappa)/D^2}$ and $\tau = \sqrt{m_v} \left(\frac{2\kappa}{\epsilon} + \frac{\epsilon k B_2}{2m_v} \right)$, we have in Algorithm 1, with probability at least $1 - p - o(1)$, that $\widehat{S} = S$ and for any $\gamma \geq 0$, for each $l \in S$:*

$$\| \phi_{\text{est},l} - \phi_l \|_{L^\infty[-1,1]} \leq [59(1 + \gamma)] \left(\frac{4C\sqrt{m_v}\kappa}{\epsilon} + \frac{C\epsilon k B_2}{\sqrt{m_v}} \right) + \frac{87}{64m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]}. \tag{5.2}$$

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[8] I_l is the "critical" interval defined in Assumption 3 for detecting $l \in S$.

Note that we query f now $N|\mathcal{X}|(|\mathcal{V}|+1)$ times. Also, $|\mathcal{X}| = (2m_x + 1) = \Theta(1)$, and $\kappa = O(k^{-1})$, as D, C, B_2, δ are constants. Hence the choice $|\mathcal{V}| = O(k \log d)$ gives us $N = O(k^2 \log(p^{-1}k^2 \log d))$ and leads to an overall query complexity of: $O(k^3 \log d \log(p^{-1}k^2 \log d))$ when the samples are corrupted with additive Gaussian noise. Choosing $p = O(d^{-c})$ for any constant $c > 0$ gives us a sample complexity of $O(k^3 (\log d)^2)$, and ensures that the result holds with high probability. The $o(1)$ term goes to zero exponentially fast as $d \to \infty$.

Simulation results. We now provide simulation results on synthetic data to support our theoretical findings. We consider the noisy setting with the point queries being corrupted with Gaussian noise. For $d = 1000, k = 4$ and $S = \{2, 105, 424, 782\}$, consider $f : \mathbb{R}^d \to \mathbb{R}$ where $f = \phi_2(x_2) + \phi_{105}(x_{105}) + \phi_{424}(x_{424}) + \phi_{782}(x_{782})$ with: $\phi_2(x) = \sin(\pi x)$, $\phi_{105}(x) = \exp(-2x)$, $\phi_{424}(x) = (1/3)\cos^3(\pi x) + 0.8x^2$, $\phi_{782}(x) = 0.5x^4 - x^2 + 0.8x$. We choose $\delta = 0.3, D = 0.2$ which can be verified as valid parameters for the above ϕ_l's. Furthermore, we choose $m_x = \lceil 2/\delta \rceil = 7$ and $m_v = \lceil 2k \log d \rceil = 56$ to satisfy the conditions of Theorem 4. Next, we choose constants $C = 0.2, B_2 = 35$ and $\kappa = 0.95 \frac{D^2}{16C^2 k B_2} = 4.24 \times 10^{-4}$ as required by Theorem 4. For the choice $\epsilon = \frac{D\sqrt{m_v}}{2CkB_2} = 0.0267$, we then query f at $(2m_x + 1)(m_v + 1) = 855$ points. The function values are corrupted with Gaussian noise: $\mathcal{N}(0, \sigma^2/N)$ for $\sigma = 0.01$ and $N = 100$. This is equivalent to resampling and averaging the points queries N times. Importantly the sufficient condition on N, as stated in Theorem 4 is $\lceil \frac{\sigma^2}{\kappa^2} \log(\frac{\sqrt{2}\sigma|\mathcal{X}||\mathcal{V}|}{\kappa p}) \rceil = 6974$ for $p = 0.1$. Thus we consider a significantly *undersampled* regime. Lastly we select the threshold $\tau = \sqrt{m_v}\left(\frac{2\kappa}{\epsilon} + \frac{\epsilon k B_2}{2m_v}\right) = 0.2875$ as stated by Theorem 4, and employ Algorithm 1 for different values of the smoothing parameter γ.

(a) Estimates of ϕ_2 (b) Estimates of ϕ_{105} (c) Estimates of ϕ_{424} (d) Estimates of ϕ_{782}
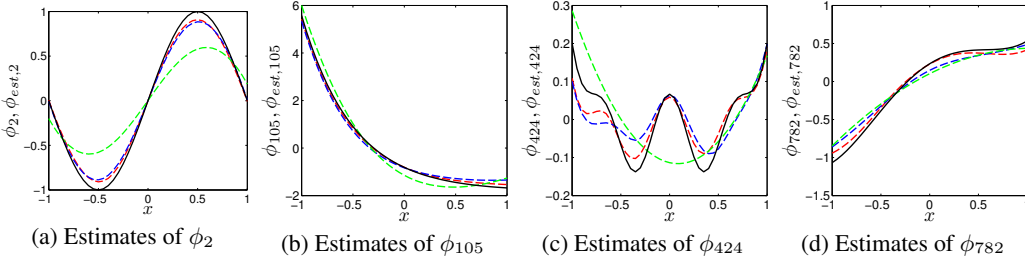
Figure 2: Estimates $\phi_{\text{est},l}$ of ϕ_l (black) for: $\gamma = 0.3$ (red), $\gamma = 1$ (blue) and $\gamma = 5$ (green).

The results are shown in Figure 2. Over 10 independent runs of the algorithm we observed that S was recovered exactly each time. Furthermore we see from Figure 2 that the recovery is quite accurate for $\gamma = 0.3$. For $\gamma = 1$ we notice that the search interval $\gamma\tau = 0.2875$ becomes large enough so as to cause the estimates $\phi_{\text{est},424}, \phi_{\text{est},782}$ to become relatively smoother. For $\gamma = 5$, the search interval $\gamma\tau = 1.4375$ becomes wide enough for a line to fit in the feasible region for ϕ'_{424}, ϕ'_{782}. This results in $\phi_{\text{est},424}, \phi_{\text{est},782}$ to be quadratic functions. In the case of ϕ'_2, ϕ'_{105}, the search interval is not sufficiently wide enough for a line to lie in the feasible region, even for $\gamma = 5$. However we notice that the estimates $\phi_{\text{est},2}, \phi_{\text{est},105}$ become relatively smoother as expected.

6 Conclusion

We proposed an efficient sampling scheme for learning SPAMs. In particular, we showed that with only a few queries, we can derive uniform approximations to each underlying univariate function of the SPAM. A crucial component of our approach is a novel convex QP for robust estimation of univariate functions via cubic splines, from samples corrupted with arbitrary bounded noise. Lastly, we showed how our algorithm can handle noisy point queries for both (i) arbitrary bounded and (ii) i.i.d. Gaussian noise models. An important direction for future work would be to determine the optimality of our sampling bounds by deriving corresponding lower bounds on the sample complexity.

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Supplementary Material : Efficient Sampling for Learning Sparse Additive Models in High Dimensions.

In this supplementary material, we prove the results stated in the paper. We first provide a collection of general results along with proofs in Section A. which are used in order to prove our main results. Section B contains the proof of Lemma 1 while in Section C we prove Theorem 2. Section D contains the proof of our main theorem for estimating in the the absence of external noise in point queries, i.e. Theorem 1. In Section E we prove Theorem 3, which is concerned with the setting where the point queries are corrupted by arbitrary bounded noise. In Section F we prove Theorem 4 that handles the setting of stochastic noise corrupting the point queries.

A   Some general results

The following Proposition is well known.

Proposition 1. *Let* $g \in C^2[x_i, x_{i+1}]$ *be such that* $g(x_i) = g(x_{i+1}) = 0$. *We then have that* $\| g \|_\infty \leq (1/8) \| g^{(2)} \|_\infty h^2$ *where* $h := x_{i+1} - x_i$.

Proof. Consider $e(x) = g(x) - w(t)(x - x_i)(x - x_{i+1})$ where for any fixed $t \in (x_i, x_{i+1})$, $w(t)$ is chosen so that

$$w(t) = \frac{g(t)}{(t - x_i)(t - x_{i+1})}. \tag{A.1}$$

Since $t \neq x_i, x_{i+1}$, $w(t)$ is well defined. Hence $e(x_i) = e(x_{i+1}) = e(t) = 0$. By repeated application of Rolle's theorem, this then implies that $\exists \zeta \in (x_i, x_{i+1})$ such that $e''(\zeta) = 0$. But $e''(x) = g''(x) - 2w(t)$, hence $w(t) = g''(\zeta)/2$. Lastly we note that $|(t - x_i)(t - x_{i+1})| \leq \frac{h^2}{4}$. Lastly, since $e(t) = 0$ we obtain

$$g(t) = \frac{g''(\zeta)}{2}(t - x_i)(t - x_{i+1}) \Rightarrow |g(t)| \leq \frac{\| g'' \|_\infty h^2}{8}. \tag{A.2}$$

$\square$

The following lemma is a simplified version of Lemma 1 of "T.Lucas, Error Bounds for Interpolating Cubic Splines Under Various End Conditions,SIAM Journal on Num. Anal.,1974".

Lemma 2. *Let* $f \in C^4[t_1, t_2]$ *and let* $\mathcal{L}$ *be a cubic spline interpolate of* f *on* $[t_1, t_2]$ *on the mesh* $(t_1 =)x_0 < x_1 < \cdots < x_n(= t_2)$ *with* $x_i - x_{i-1} = h$ *for* $i = 1, \ldots, n$. *If*

$$\max_{0 \leq i \leq n} |f_i'' - \mathcal{L}_i''| \leq Kh^2 \tag{A.3}$$

holds for some constant $K > 0$ *then it implies that :*

$$\| f - \mathcal{L} \|_\infty \leq \frac{h^4}{8} \left(K + \frac{\| f^{(4)} \|_\infty}{8} \right). \tag{A.4}$$

Proof. Since $\mathcal{L}''$ is linear for $x \in [x_i, x_{i+1}]$ we have:

$$\mathcal{L}''(x) = \frac{x - x_i}{h} \mathcal{L}_{i+1}'' + \frac{(x_{i+1} - x)}{h} \mathcal{L}_i''; \quad x \in [x_i, x_{i+1}]. \tag{A.5}$$

Similarly we obtain

$$f''(x) = \underbrace{\frac{x - x_i}{h} f_{i+1}'' + \frac{(x_{i+1} - x)}{h} f_i''}_{l_{f''}(x)} + R_i(x); \quad x \in [x_i, x_{i+1}] \tag{A.6}$$

where $l_{f''}(x)$ is the linear intepolant of f'' between x_i and x_{i+1}. Since $f'' - l_{f''} \in C^2[x_i, x_{i+1}]$ and $f''(x_i) - l_{f''}(x_i) = f''(x_{i+1}) - l_{f''}(x_{i+1}) = 0$, therefore by Proposition 1 we have that $|R_i(x)| \leq (1/8)h^2 \| f^{(4)} \|_\infty$. Subtracting (A.5) from (A.6) we then obtain

$$\parallel f'' - \mathcal{L}'' \parallel_\infty \ \leq \ \max_{0 \leq i \leq n} |f_i'' - \mathcal{L}_i''| + \frac{1}{8}h^2 \parallel f^{(4)} \parallel_\infty, \tag{A.7}$$

$$\leq Kh^2 + \frac{1}{8}h^2 \parallel f^{(4)} \parallel_\infty = h^2 \left(K + \frac{\parallel f^{(4)} \parallel_\infty}{8} \right). \tag{A.8}$$

Since $f - \mathcal{L} \in C^2[t_1, t_2]$ and $f(x_i) - \mathcal{L}(x_i) = f(x_{i+1}) - \mathcal{L}(x_{i+1}) = 0$ for $i = 0, \ldots, n-1$ we then have from Proposition 1 that:

$$\parallel f - \mathcal{L} \parallel_\infty \leq \frac{h^2}{8} \parallel f'' - \mathcal{L}'' \parallel_\infty \leq \frac{h^4}{8} \left(K + \frac{\parallel f^{(4)} \parallel_\infty}{8} \right).$$

$\square$

Proposition 2 is a standard result, see for example Lemma 4 of "T.Lucas, Error Bounds for Interpolating Cubic Splines Under Various End Conditions,SIAM Journal on Num. Anal.,1974".

Proposition 2. *For the tridiagonal, symmetric $m \times m$ matrix*

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{pmatrix},$$

we have that $\mathbf{A}$ is invertible and $\parallel \mathbf{A}^{-1} \parallel_\infty \leq 1/2$.

Proof. The fact that $\mathbf{A}$ is invertible follows from the fact that it is strictly diagonally dominant. Now note that $\mathbf{A} = 4\mathbf{I} + \mathbf{B}$ where $\parallel \mathbf{B} \parallel_\infty \leq 2$. For any $\mathbf{x} \in \mathbb{R}^m$ such that $\parallel \mathbf{x} \parallel_\infty = 1$ we have:

$$\parallel \mathbf{Ax} \parallel_\infty \geq 4 - \parallel \mathbf{Bx} \parallel_\infty \geq 4 - \parallel \mathbf{B} \parallel_\infty \geq 2. \tag{A.9}$$

Now for any $\mathbf{x}$, $\parallel \mathbf{x} \parallel_\infty = 1$; we have $\parallel \mathbf{A}^{-1}\mathbf{Ax} \parallel_\infty = 1$ which implies

$$\parallel \mathbf{A}^{-1} \left(\frac{\mathbf{Ax}}{\parallel \mathbf{Ax} \parallel_\infty} \right) \parallel_\infty = \frac{1}{\parallel \mathbf{Ax} \parallel_\infty} \leq (1/2).$$

$\square$

Proposition 3. 1. *Let $g \in C^4[x - h_0, x + h_0]$, $h_0 > 0$. Then $\forall h \in (0, h_0)$:*

$$\left| \frac{g(x+h) - 2g(x) + g(x-h)}{h^2} - g'(x) \right| \leq \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty . \tag{A.10}$$

2. *Let $g \in C^2[x - h_0, x + h_0]$, $h_0 > 0$. Then $\forall h \in (0, h_0)$, $\exists \zeta \in (x - h_0, x + h_0)$ such that*

$$g(x+h) - 2g(x) + g(x-h) = h^2 g''(\zeta). \tag{A.11}$$

Proof. 1. We have by Taylor's expansion:

$$g(x+h) = g(x) + hg'(x) + \frac{h^2}{2}g''(x) + \frac{h^3}{6}g'''(x) + \frac{h^4}{24}g^{(4)}(\zeta_1),$$

$$g(x-h) = g(x) - hg'(x) + \frac{h^2}{2}g''(x) - \frac{h^3}{6}g'''(x) + \frac{h^4}{24}g^{(4)}(\zeta_2),$$

where $\zeta_1 \in (x, x+h)$ and $\zeta_2 \in (x-h, x)$. This then gives us:

$$\frac{g(x+h) - 2g(x) + g(x-h)}{h^2} = \frac{h^2 g''(x) + \frac{h^4}{24}g^{(4)}(\zeta_1) + \frac{h^4}{24}g^{(4)}(\zeta_2)}{h^2}$$

$$= g''(x) + \frac{h^2}{12}g^{(4)}(\zeta)$$

for some $\zeta \in (x - h_0, x + h_0)$. The second equality above follows from continuity of $g^{(4)}$ in $[x - h_0, x + h_0]$. This completes the proof.

3

2. We have by Taylor's expansion:

$$g(x + h) = g(x) + hg'(x) + \frac{h^2}{2}g''(\zeta_1),$$

$$g(x - h) = g(x) - hg'(x) + \frac{h^2}{2}g''(\zeta_2),$$

where $\zeta_1 \in (x, x+h)$ and $\zeta_2 \in (x-h, x)$. This then gives us for some $\zeta \in (x-h_0, x+h_0)$:

$$g(x + h) - 2g(x) + g(x - h) = h^2 \frac{g''(\zeta_1) + g''(\zeta_2)}{2} = h^2 g''(\zeta)$$

where the last equality follows from continuity of g''.

$\square$

Lemma 3. *Let $f \in C^4[t_1, t_2]$ and let $(t_1 =)\, x_0 < x_1 < \cdots < x_n(= t_2)$ be a uniform mesh with $x_i - x_{i-1} = h$ for $i = 1, \ldots, n$. Furthermore let*

$$R_i := (f''_{i-1} + 4f''_i + f''_{i+1}) - \frac{6}{h^2}(f_{i-1} - 2f_i + f_{i+1}); \ i = 1, \ldots, n - 1. \qquad \text{(A.12)}$$

We then have $|R_i| \leq \frac{3h^2}{2} \parallel f^{(4)} \parallel_\infty, \forall i.$

Proof. From Proposition 3(1) we have $\frac{f_{i-1} - 2f_i + f_{i+1}}{h^2} = f''_i + e_i$ where $|e_i| \leq (h^2/12) \parallel f^{(4)} \parallel_\infty$ for $i = 1, \ldots, n - 1$. This then gives us:

$$R_i = (f''_{i-1} - 2f''_i + f''_{i+1}) - 6e_i = h^2 f^{(4)}(\zeta_i) - 6e_i$$

where the last equality follows from Proposition 3(2) for some $\zeta_i \in (x_{i-1}, x_{i+1})$. Hence we obtain $|R_i| \leq h^2 \parallel f^{(4)} \parallel_\infty + 6|e_i| \leq \frac{3h^2}{2} \parallel f^{(4)} \parallel_\infty$ for all $i = 1, \ldots, n - 1$. $\square$

A slightly generalized version of Proposition 4 was proven in Lemma 2 of "R.Beatson, On the Convergence of Some Cubic Spline Interpolation Schemes, SIAM Journal on Num. Anal.,1986", however the constants were not derived explicitly. In the following, we explicitly state the constants appearing in the bounds.

Proposition 4. *Consider the $(n + 1) \times (n + 1)$ matrix:*

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & -2 & 1 \end{pmatrix}.$$

We have that $\mathbf{A}$ is invertible and also that $\parallel \mathbf{A}^{-1} \parallel_\infty \leq (7/3)$.

Proof. For some $\mathbf{c} = (c_0 \ c_1 \ \cdots \ c_{n-1} \ c_n) \in \mathbb{R}^{n+1}$ consider the linear system $\mathbf{Ax} = \mathbf{c}$ where $\mathbf{x} = (x_0 \ x_1 \ \cdots \ x_{n-1} \ x_n)$. Next, consider the first 3 rows (denoted R_1, R_2, R_3 from top to bottom) of the augmented matrix $[\mathbf{A} : \mathbf{c}]$:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & \cdots & 0 & \vdots & c_0 \\ 1 & 4 & 1 & 0 & \cdots & \cdots & 0 & \vdots & c_1 \\ 0 & 1 & 4 & 1 & \cdots & \cdots & 0 & \vdots & c_2 \end{pmatrix}.$$

Perform row operation $R_2 \to R_2 - R_1$ followed by $R_3 \to R_3 - (1/6)R_2$ to obtain:

$$\begin{pmatrix} 1 & -2 & 1 & 0 & \cdots & \cdots & 0 & \vdots & c_0 \\ 0 & 6 & 0 & 0 & \cdots & \cdots & 0 & \vdots & c_1 - c_0 \\ 0 & 0 & 4 & 1 & \cdots & \cdots & 0 & \vdots & c_2 - \frac{(c_1 - c_0)}{6} \end{pmatrix}.$$

4

Similar, consider the bottom 3 rows (denoted R_1, R_2, R_3 from top to bottom) of $[\mathbf{A} : \mathbf{c}]$:

$$\begin{pmatrix} 0 & \cdots & \cdots & 1 & 4 & 1 & 0 & \vdots & c_{n-2} \\ 0 & \cdots & \cdots & 0 & 1 & 4 & 1 & \vdots & c_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & \vdots & c_n \end{pmatrix}.$$

Perform row operation $R_2 \to R_2 - R_3$ followed by $R_1 \to R_1 - (1/6)R_2$ to obtain:

$$\begin{pmatrix} 0 & \cdots & \cdots & 1 & 4 & 0 & 0 & \vdots & c_{n-2} - \frac{(c_{n-1}-c_n)}{6} \\ 0 & \cdots & \cdots & 0 & 0 & 6 & 0 & \vdots & c_{n-1} - c_n \\ 0 & \cdots & \cdots & 0 & 1 & -2 & 1 & \vdots & c_n \end{pmatrix}.$$

We hence obtain the following equivalent system of linear equations:

$$\underbrace{\begin{pmatrix} 4 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 4 \end{pmatrix}}_{\mathbf{A}'} \underbrace{\begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_{n-3} \\ x_{n-2} \end{pmatrix}}_{\mathbf{x}'} = \underbrace{\begin{pmatrix} c_2 - \frac{(c_1 - c_0)}{6} \\ c_3 \\ \vdots \\ \vdots \\ c_{n-3} \\ c_{n-2} - \frac{(c_{n-1}-c_n)}{6} \end{pmatrix}}_{\mathbf{c}'}, \qquad \text{(A.13)}$$

where $\mathbf{A}' \in \mathbb{R}^{(n-1) \times (n-1)}$ is symmetric, tridiagonal,

$$6x_1 = c_1 - c_0 \quad \text{and} \quad x_0 - 2x_1 + x_2 = c_0, \qquad \text{(A.14)}$$
$$6x_{n-1} = c_{n-1} - c_n \quad \text{and} \quad x_{n-2} - 2x_{n-1} + x_n = c_n. \qquad \text{(A.15)}$$

Now $\mathbf{A}'$ is strictly diagonally dominant implying a unique solution $\mathbf{x}' = (x_2 \ \ldots \ x_{n-2})$ to the system $\mathbf{A}'\mathbf{x}' = \mathbf{c}'$. Using the obtained x_2, x_{n-2} in (A.14),(A.15) we then get a unique solution $\mathbf{x} = (x_0 \ x_1 \ \ldots \ x_{n-1} \ x_n)$ to the system $\mathbf{A}\mathbf{x} = \mathbf{c}$ implying that $\mathbf{A}$ is invertible.

We now proceed to derive an upper bound on $\| \mathbf{A}^{-1} \|_\infty$. First note from Proposition 2 that $\| \mathbf{A}'^{-1} \|_\infty \leq 1/2$. Also note that $\| \mathbf{c}' \|_\infty \leq \| \mathbf{c} \|_\infty + 2(1/6) \| \mathbf{c} \|_\infty = (4/3) \| \mathbf{c} \|_\infty$. This then implies that $\| \mathbf{x}' \|_\infty \leq \| \mathbf{A}'^{-1} \|_\infty \| \mathbf{c}' \|_\infty \leq (2/3) \| \mathbf{c} \|_\infty$. From (A.14) we obtain $|x_1| \leq \| \mathbf{c} \|_\infty /3$ and also

$$|x_0| = |2x_1 - x_2 + c_0| \leq \frac{2 \| \mathbf{c} \|_\infty}{3} + \frac{2 \| \mathbf{c} \|_\infty}{3} + \| \mathbf{c} \|_\infty \leq \frac{7}{3} \| \mathbf{c} \|_\infty$$

where we used $|x_2| \leq \| \mathbf{x}' \|_\infty \leq (2/3) \| \mathbf{c} \|_\infty$. Similarly, from (A.15), we obtain $|x_{n-1}| \leq \| \mathbf{c} \|_\infty /3$ and $|x_n| \leq \frac{7}{3} \| \mathbf{c} \|_\infty$ repsectively. These bounds collectively imply $\| \mathbf{x} \|_\infty \leq (7/3) \| \mathbf{c} \|_\infty$. Lastly, we have that $\mathbf{x} = \mathbf{A}^{-1}\mathbf{c}$ implies

$$\| \mathbf{A}^{-1}\mathbf{c} \|_\infty = \| \mathbf{x} \|_\infty \leq \frac{7}{3} \| \mathbf{c} \|_\infty \quad \Rightarrow \quad \| \mathbf{A}^{-1} \frac{\mathbf{c}}{\| \mathbf{c} \|_\infty} \|_\infty \leq \frac{7}{3}.$$

Since the above holds for all $\mathbf{c}$ we obtain $\| \mathbf{A}^{-1} \|_\infty \leq (7/3)$. $\qquad \square$

Proposition 5. *Consider the $(n-1) \times (n-1)$ matrix:*

$$\mathbf{B} = \begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}.$$

We have that $\mathbf{B}$ is invertible and $\| \mathbf{B}^{-1} \|_\infty \leq \frac{n^2}{\pi^2} \sqrt{n-1} \left(1 - \frac{\pi^2}{12} \right)^{-1}$.

Proof. It is well known (cf. "S. Noschese et al. Tridiagonal Toeplitz matrices: properties and novel applications, Numerical Linear Algebra with Applications, 2013" and references within) that eigenvalues $(\lambda_s)_{s=1}^{n-1}$ of $\mathbf{B}$ are given by $\lambda_s = -2 + 2\cos(s\pi/n)$. Clearly $\lambda_s \neq 0$ for $s = 1, \ldots, n-1$ implying $\mathbf{B}$ is invertible.

Next it is easy to see that $\| \mathbf{B}^{-1} \|_\infty \leq \| \mathbf{B}^{-1} \|_2 \sqrt{n-1}$. Indeed for any $\mathbf{x} \in \mathbb{R}^{(n-1)\times(n-1)}$:

$$\| \mathbf{B}^{-1}\mathbf{x} \|_\infty \leq \| \mathbf{B}^{-1}\mathbf{x} \|_2 \leq \| \mathbf{B}^{-1} \|_2 \| \mathbf{x} \|_\infty \sqrt{n-1}.$$

Since $\mathbf{B}$ is symmetric therefore:

$$\| \mathbf{B}^{-1} \|_\infty \leq \max_s \left| \lambda_s(\mathbf{B}^{-1}) \right| \sqrt{n-1} = \frac{1}{\min_s |\lambda_s(\mathbf{B})|} \sqrt{n-1}. \tag{A.16}$$

Now observe that

$$\min_s |\lambda_s(\mathbf{B})| = 2 \left(1 - \cos \frac{\pi}{n} \right). \tag{A.17}$$

Using Taylors remainder theorem we obtain for some $\zeta \in (0, \pi/n)$:

$$\min_s |\lambda_s(\mathbf{B})| = 2 \left(1 - \left(1 - \frac{\pi^2}{2n^2} + \frac{\pi^4}{24n^4} \cos(\zeta) \right) \right), \tag{A.18}$$

$$= \frac{\pi^2}{n^2} - \frac{\pi^4}{12n^4} \cos(\zeta) \geq \frac{\pi^2}{n^2} \left(1 - \frac{\pi^2}{12} \right). \tag{A.19}$$

$\square$

B Proof of Lemma 1

First recall, that we recover a stable approximation $\widehat{\mathbf{x}}_i$ to $\mathbf{x}_i$ via ℓ_1 minimization [17, 18] as follows:

$$\widehat{\mathbf{x}}_i := \triangle(\mathbf{y}_i) := \operatorname*{argmin}_{\mathbf{y}_i = \mathbf{V}\mathbf{z}} \| \mathbf{z} \|_1 . \tag{B.1}$$

Now in order to prove this lemma, we make use of a key theorem from [8]. While the first part is by now standard (see for example "R. Baraniuk et al., A Simple Proof of the Restricted Isometry Property for Random Matrices, Constructive Approximation, 2008"), the second result was stated in [8] as a specialization of Theorem 1.2 from [19] to the case of Bernoulli measurement matrices.

Theorem 5 ([19, 8])**.** *Let $\mathbf{V}$ be a $m \times d$ random matrix with all entries being Bernoulli i.i.d random variables scaled with $1/\sqrt{m}$. Then the following results hold.*

1. *Let $0 < \mu < 1$. Then there are two positive constants $c_1, c_2 > 0$, such that the matrix $\mathbf{V}$ has the Restricted Isometry Property*

$$(1 - \mu) \| \mathbf{x} \|_2^2 \leq \| \mathbf{V}\mathbf{x} \|_2^2 \leq (1 + \mu) \| \mathbf{x} \|_2^2 \tag{B.2}$$

   for all $\mathbf{x} \in \mathbb{R}^d$ such that $\#supp(\mathbf{x}) \leq c_2 m / \log(d/m)$ with probability at least $1 - e^{-c_1 m}$.

2. *Let us suppose $d > (\log 6)^2 m$. Then there are positive constants $C, c_1', c_2' > 0$ such that with probability at least $1 - e^{-c_1' m} - e^{-\sqrt{md}}$ the matrix $\mathbf{V}$ has the following property. For every $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{n} \in \mathbb{R}^m$ and every natural number $K \leq c_2' m / \log(d/m)$ we have*

$$\| \triangle(\mathbf{V}\mathbf{x} + \mathbf{n}) - \mathbf{x} \|_2 \leq C \left(K^{-1/2} \sigma_K(\mathbf{x})_1 + \max \left\{ \| \mathbf{n} \|_2, \sqrt{\log d} \| \mathbf{n} \|_\infty \right\} \right), \tag{B.3}$$

   where

$$\sigma_K(\mathbf{x})_1 := \inf \left\{ \| \mathbf{x} - \mathbf{z} \|_1 \colon \#supp(\mathbf{z}) \leq K \right\}$$

   is the best K-term approximation of $\mathbf{x}$.

Remark 1. *The proof of the second part of Theorem 5 requires (B.2) to hold, which is the case in our setting with high probability.*

Applying Theorem 5 to our setting we obtain the following corollary that bounds $\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2$ for all $i = -m_x, \ldots, m_x$ with high probability. This proves the first part of Lemma 1.

Corollary 2. *There exist constants $c'_3 \geq 1$ and $C, c'_1 > 0$ such that for m_v satisfying $c'_3 k \log d < m_v < d/(\log 6)^2$ we have with probability at least $1 - e^{-c'_1 m_v} - e^{-\sqrt{m_v d}}$ that $\widehat{\mathbf{x}}_i$ as obtained in (B.1) satisfies for all $i = -m_x, \ldots, m_x$:*

$$\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2 \leq \frac{C\epsilon k B_2}{2\sqrt{m_v}}, \tag{B.4}$$

where $B_2 > 0$ is the constant defined in (2.1).

Proof. We first note that $\mathbf{x}_i$ is at most k-sparse for each $i = -m_x, \ldots, m_x$ implying $\sigma_k(\mathbf{x}_i)_1 = 0$. This gives us:

$$\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2 \leq C \max \left\{ \| \mathbf{n}_i \|_2, \sqrt{\log d} \| \mathbf{n}_i \|_\infty \right\}; \ i = -m_x, \ldots, m_x. \tag{B.5}$$

Recall that $\mathbf{n}_i = [n_{i,1} \ldots n_{i,m_v}]$ where $n_{i,j} = \frac{1}{2}\epsilon \mathbf{v}_j^T \triangledown^2 f(\zeta_{i,j})\mathbf{v}_j$. Since $f(\zeta_{i,j}) = \sum_{l \in S} \phi_l(\zeta_{i,j}^{(l)})$ therefore $\triangledown^2 f(\zeta_{i,j})$ is a diagonal matrix with the non-zero entries being $[\triangledown^2 f(\zeta_{i,j})]_{l,l} = \phi_l''(\zeta_{i,j}^{(l)})$. Hence we have for all $i = -m_x, \ldots, m_x$ that

$$n_{i,j} = \frac{\epsilon}{2}\sum_{l \in S} v_{j,l} \phi_l''(\zeta_{i,j}^{(l)}) v_{j,l} \Rightarrow |n_{i,j}| \leq \frac{\epsilon}{2}\sum_{l \in S} \left| \phi_l''(\zeta_{i,j}^{(l)}) \right| v_{j,l}^2 \leq \frac{\epsilon k B_2}{2m_v}$$

holds which then implies that $\| \mathbf{n}_i \|_2 = (\sum_{j=1}^{m_v} |n_{i,j}|^2)^{1/2} \leq \frac{\epsilon k B_2}{2\sqrt{m_v}}$. Plugging this in (B.5) we get for the stated choice of m_v that:

$$\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2 \leq C \max \left\{ \frac{\epsilon k B_2}{2\sqrt{m_v}}, \sqrt{\log d}\frac{\epsilon k B_2}{2m_v} \right\},$$

$$= \frac{C\epsilon k B_2}{2\sqrt{m_v}} \max \left\{ 1, \frac{\sqrt{\log d}}{\sqrt{m_v}} \right\} = \frac{C\epsilon k B_2}{2\sqrt{m_v}}.$$

$\square$

From Corollary 2 we observe that the decoding rule (B.1), for an appropriate choice of m_v, provides us with estimates: $\widehat{\phi}'_l(i/m_x)$ of $\phi'_l(i/m_x)$ for $i = -m_x, \ldots, m_x$ and for each $l = 1, \ldots, d$. We let $\tau = \frac{C\epsilon k B_2}{2\sqrt{m_v}}$ denote the error bound of Corollary 2. Recall that we consider the following estimate to S:

$$\widehat{S} := \cup_{i=-m_x}^{m_x} \left\{ l \in \{1, \ldots, d\} : \left| \widehat{\phi}'_l(i/m_x) \right| > \tau \right\}. \tag{B.6}$$

The following Lemma gives precise conditions under which $\widehat{S} = S$ is guaranteed to hold. This proves the second part of Lemma 1.

Lemma 4. *Let m_x be chosen such that $m_x \geq 1/\delta$ holds true with δ as defined in Assumption 3. Assuming that the conditions of Corollary 2 are satisfied, we have for any $\epsilon < \frac{D\sqrt{m_v}}{Ck B_2}$ that $\widehat{S} = S$ holds true.*

Proof. Firstly observe that for all $i = -m_x, \ldots, m_x$ and $l = 1, \ldots, d$:

$$\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2 \leq \tau \Rightarrow \widehat{\phi}'_l(i/m_x) \in [\phi'_l(i/m_x) - \tau, \phi'_l(i/m_x) + \tau]. \tag{B.7}$$

Therefore for all $l \notin S$ we will have that $\widehat{\phi}'_l(i/m_x) \in [-\tau, \tau], \forall i$. This means that we can recover all those $l \in S$ for which $\exists i \in \{-m_x, \ldots, m_x\}$ such that $\left| \widehat{\phi}'_l(i/m_x) \right| > \tau$ holds. Crucially, as a consequence of Assumption 3, we have for the choice $m_x \geq 1/\delta$, that for each $l \in S$, $\exists i \in \{-m_x, \ldots, m_x\}$ such that $|\phi'_l(i/m_x)| \geq D$ holds. Thus if $\tau < D/2$, then for all $l \in S$, $\left| \widehat{\phi}'_l(i/m_x) \right| > \tau$ for at least one i. Lastly, $\tau < D/2$ is ensured for the stated choice of ϵ. $\square$

C   Proof of Theorem 2

Before proceeding, we need to prove some secondary results that are used in the proof of Theorem 2. We begin in Section C.1 by first defining cubic splines formally and provide an approximation result for the not-a-knot cubic spline, in the noise-less setting. We note that this result is of independent interest. Next, in Section C.2 we provide an equivalent representation of (P) in terms of the coefficients of the cubic spline. Lastly, in Section C.3 we provide the proof of Theorem 2.

C.1   Cubic splines and the not-a-knot cubic spline

We first begin by giving an overview of cubic spline interpolation in the setting where the exact values of the function (to be interpolated) are available. More formally let $t_1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = t_2$ be a uniform mesh[9] where $x_{i+1} - x_i = h$. A spline of degree p is a function $\mathcal{L}(x)$ which satisfies the following conditions:

1. For $x \in [x_i, x_{i+1}]$, $\mathcal{L}(x) = \mathcal{L}_i(x)$ (polynomial of degree at most p).
2. $\mathcal{L}^{(j)}$ exists and is continuous at the interior points $x_1, \ldots, x_{n-1}$ for all $1 \leq j \leq p-1$.

Let $f \in C^4[t_1, t_2]$ be the function to be interpolated and say we are given $f_0, f_1, \ldots, f_n$ where $f_i = f(x_i)$. Our aim is to find a cubic spline : $\mathcal{L}$ such that $\mathcal{L}(x_i) = f_i$ for $i = 0, \ldots, n$. Hence in $[x_i, x_{i+1}]$, $\mathcal{L}_i(x)$ is a cubic polynomial such that the following conditions are satisfied.

1. $\mathcal{L}_i(x_i) = f_i$ and $\mathcal{L}_i(x_{i+1}) = f_{i+1}$ for $i = 0, \ldots, n-1$. **(Interpolation condition)**
2. $\mathcal{L}'_i(x_{i+1}) = \mathcal{L}'_{i+1}(x_{i+1})$ for $i = 0, \ldots, n-2$. **(Continuity of $\mathcal{L}'$ at $x_1, \ldots, x_{n-1}$)**
3. $\mathcal{L}''_i(x_{i+1}) = \mathcal{L}''_{i+1}(x_{i+1})$ for $i = 0, \ldots, n-2$. **(Continuity of $\mathcal{L}''$ at $x_1, \ldots, x_{n-1}$)**

From (3) we have $\mathcal{L}''_i(x_i) = a_i = \mathcal{L}''_{i-1}(x_i)$. We can write $\mathcal{L}''_i$ as a Lagrange first order interpolating polynomial that interpolates a_i and a_{i+1} between x_i and x_{i+1} respectively. By integrating twice we have the following form for $\mathcal{L}_i(x)$:

$$\mathcal{L}_i(x) = \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} + b_i(x_{i+1} - x) + c_i(x - x_i). \tag{C.1}$$

Finding $\mathcal{L}$ is hence equivalent to finding the $(3n + 1)$ variables:

$$\mathbf{a} = (a_0, a_1, \ldots, a_n), \mathbf{b} = (b_0, \ldots, b_{n-1}), \mathbf{c} = (c_0, \ldots, c_{n-1}). \tag{C.2}$$

Note that Condition 1 involves $2n$ constraints while Condition 2 results in $n-1$ constraints. Hence the above conditions give rise to $3n-1$ equations leaving 2 free variables. There are several ways of taking care of this by introducing constraints at the boundary intervals (see [25] for examples).

Not-a-knot cubic splines.   One popular boundary condition was introduced in [25] and gives rise to the so-called "not-a-knot" cubic spline. This boundary condition does away with the requirement of having apriori knowledge about the derivatives of f at the boundary points x_0, x_n. The idea here is to *not* treat x_1, x_{n-1} as knots (hence the name) implying that there is a single cubic polynomial in the intervals $[x_0, x_2]$ and $[x_{n-2}, x_n]$ respectively. As each $\mathcal{L}_i$ is a cubic polynomial, this is equivalent to the boundary conditions:

$$\mathcal{L}'''_0(x_1) = \mathcal{L}'''_1(x_1) \text{ and } \mathcal{L}'''_{n-2}(x_{n-1}) = \mathcal{L}'''_{n-1}(x_{n-1}). \tag{C.3}$$

It is known that if $f \in C^4[t_1, t_2]$, then the approximation error is bounded[10] from above by $(40/64) \parallel f^{(4)} \parallel_\infty h^4$. It appears to the best of our knowledge, that the optimal error constant for this class of cubic splines is not known. We present in the form of the following proposition an improved error [11] bound of $(29/64) \parallel f^{(4)} \parallel_\infty h^4$ for uniform meshes. The proof technique is similar to that in "R.Beatson, On the Convergence of Some Cubic Spline Interpolation Schemes, SIAM Journal on Num. Anal.,1986". However the results there are for C^2, C^3 smooth f.

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[9]For ease of exposition we will restrict ourselves to uniform meshes in this paper. However one can consider a non-uniform mesh too.

[10]Actually "C. de Boor, Convergence of cubic spline interpolation with the not-a-knot condition,Technical report,1984" in which this was proven contains a bound for general meshes (not necessarily uniform.)

[11]The constant $29/64$ can most likely be improved using the integral form of Taylors remainder term.

Proposition 6. *Consider $f \in C^4[t_1, t_2]$ and let $\mathcal{L}$ be the unique cubic spline obtained using the not-a-knot boundary conditions (C.3). We then have that:*

$$\| \mathcal{L} - f \|_\infty \leq \frac{29}{64} \| f^{(4)} \|_\infty h^4. \tag{C.4}$$

Proof. Recall from (C.1) that the cubic spline $\mathcal{L}$ has the following form in the interval $[x_i, x_{i+1}]$:

$$\mathcal{L}_i(x) = \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} + b_i(x_{i+1} - x) + c_i(x - x_i).$$

Now we have for any cubic spline the following well known identity (see [20] for instance):

$$\mathcal{L}''_{i-1} + 4\mathcal{L}''_i + \mathcal{L}''_{i+1} = \frac{6}{h^2}(\mathcal{L}_{i-1} - 2\mathcal{L}_i + \mathcal{L}_{i+1}); \quad 1 \leq i \leq n - 1. \tag{C.5}$$

Recall the not-a-knot boundary conditions of (C.3):

$$\mathcal{L}'''_0(x_1) = \mathcal{L}'''_1(x_1) \quad \text{and} \quad \mathcal{L}'''_{n-2}(x_{n-1}) = \mathcal{L}'''_{n-1}(x_{n-1}).$$

Using (C.1) in (C.3) we equivalently obtain:

$$\mathcal{L}''_0 - 2\mathcal{L}''_1 + \mathcal{L}''_2 = 0 \quad \text{and} \quad \mathcal{L}''_{n-2} - 2\mathcal{L}''_{n-1} + \mathcal{L}''_n = 0. \tag{C.6}$$

(C.5) and (C.6) can then be written compactly as:

$$\underbrace{\begin{pmatrix} 1 & -2 & 1 & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & -2 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} \mathcal{L}''_0 \\ \mathcal{L}''_1 \\ \vdots \\ \vdots \\ \mathcal{L}''_{n-1} \\ \mathcal{L}''_n \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} 0 \\ f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \\ 0 \end{pmatrix}. \tag{C.7}$$

For any smooth $f \in C^4[t_1, t_2]$ we will then have the following linear system:

$$\underbrace{\begin{pmatrix} 1 & -2 & 1 & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & -2 & 1 \end{pmatrix}}_{\mathbf{A}} \begin{pmatrix} f''_0 \\ f''_1 \\ \vdots \\ \vdots \\ f''_{n-1} \\ f''_n \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} 0 \\ f_0 - 2f_1 + f_2 \\ \vdots \\ \vdots \\ f_{n-2} - 2f_{n-1} + f_n \\ 0 \end{pmatrix} + \begin{pmatrix} R_0 \\ R_1 \\ \vdots \\ \vdots \\ R_{n-1} \\ R_n \end{pmatrix}. \tag{C.8}$$

Subtracting (C.7) from (C.8) it is easy to verify with the help of Proposition 4 and Proposition 3 that

$$\max_{0 \leq i \leq n} |\mathcal{L}''_i - f''_i| \leq \| \mathbf{A}^{-1} \|_\infty \max_{0 \leq i \leq n} |R_i| \leq \frac{7}{3} \left(\frac{3h^2 \| f^{(4)} \|_\infty}{2} \right) = \frac{7h^2 \| f^{(4)} \|_\infty}{2}. \tag{C.9}$$

We used Proposition 3(2) to obtain $|R_0|, |R_n| \leq h^2 \| f^{(4)} \|_\infty$. From Proposition 3(1) we have $|R_i| \leq \frac{3h^2 \| f^{(4)} \|_\infty}{2}$ for $1 \leq i \leq n - 1$, which gives a uniform bound on $|R_i|$ for $0 \leq i \leq n$.

Lastly, Lemma 2 gives us the final error bound:

$$\| \mathcal{L} - f \|_\infty \leq \frac{h^4}{8} \left(\frac{7 \| f^{(4)} \|_\infty}{2} + \frac{1}{8} \| f^{(4)} \|_\infty \right) \leq \frac{29 \| f^{(4)} \|_\infty h^4}{64}.$$

$\square$

C.2 An equivalent representation for (P)

The following lemma restates (P) in terms of the spline coefficients mentioned in (C.2).

Lemma 5. *We have the following representation of (P) in terms of the spline coefficients:* $\mathbf{a} = (a_0, a_1, \ldots, a_n), \mathbf{b} = (b_0, \ldots, b_{n-1}),$ *and* $\mathbf{c} = (c_0, \ldots, c_{n-1})$.

$$
(P)\begin{cases}
\min \dfrac{h}{6}\mathbf{a}^T \begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} \mathbf{a} & \text{(C.10)} \\[3em]
\text{s.t.} \quad \widehat{g}_i - \gamma\tau \le b_i h + \dfrac{a_i h^2}{6} \le \widehat{g}_i + \gamma\tau; \quad i = 0, \ldots, n-1, & \text{(C.11)} \\[1.2em]
\widehat{g}_n - \gamma\tau \le c_{n-1}h + \dfrac{a_n h^2}{6} \le \widehat{g}_n + \gamma\tau, & \text{(C.12)} \\[1.2em]
c_i = b_{i+1}; \quad i = 0, \ldots, n-2, & \text{(C.13)} \\[0.5em]
a_{i+1}h = (b_i - c_i) + (c_{i+1} - b_{i+1}); \quad i = 0, \ldots, n-2, & \text{(C.14)} \\[0.5em]
a_0 - 2a_1 + a_2 = 0 \quad \text{and} \quad a_{n-2} - 2a_{n-1} + a_n = 0. & \text{(C.15)}
\end{cases}
$$

Proof. Recall from (C.1) that $\mathcal{L}(x) = \mathcal{L}_i(x)$ for $x \in [x_i, x_{i+1}]$ where

$$
\mathcal{L}_i(x) = \frac{a_i(x_{i+1} - x)^3}{6h} + \frac{a_{i+1}(x - x_i)^3}{6h} + b_i(x_{i+1} - x) + c_i(x - x_i).
$$

Now we have that $\int_{t_1}^{t_2} \mathcal{L}''(x)^2 dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (\mathcal{L}_i''(x))^2 dx$. The following is easily verified.

$$
\int_{x_i}^{x_{i+1}} (\mathcal{L}_i''(x))^2 dx = a_i^2 \frac{h}{3} + a_{i+1}^2 \frac{h}{3} + a_i a_{i+1} \frac{h}{3}. \tag{C.16}
$$

This results in

$$
\int_{t_1}^{t_2} \mathcal{L}''(x)^2 dx = \sum_{i=0}^{n-1} \left(a_i^2 \frac{h}{3} + a_{i+1}^2 \frac{h}{3} + a_i a_{i+1} \frac{h}{3} \right) \tag{C.17}
$$

$$
= \frac{h}{3} \left(a_0^2 + a_n^2 + 2(a_1^2 + \cdots + a_{n-1}^2) + \sum_{i=0}^{n-1} a_i a_{i+1} \right) \tag{C.18}
$$

$$
= \frac{h}{6} \left(2a_0^2 + 2a_n^2 + 4\sum_{i=1}^{n-1} a_i^2 + 2\sum_{i=0}^{n-1} a_i a_{i+1} \right) \tag{C.19}
$$

$$
= \frac{h}{6}\mathbf{a}^T \begin{pmatrix} 2 & 1 & & & & \\ 1 & 4 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} \mathbf{a} \tag{C.20}
$$

where $\mathbf{a} = (a_0\ a_1 \cdots a_{n-1}\ a_n)$. Note that the above matrix is strictly diagonally dominant and is also in fact positive definite. We now derive the constraints of the problem:

1. $\widehat{\mathbf{g}}_\mathbf{i} - \gamma\tau \le \mathcal{L}(\mathbf{x_i}) \le \widehat{\mathbf{g}}_\mathbf{i} + \gamma\tau; \quad \mathbf{i} = \mathbf{0}, \ldots, \mathbf{n}.$

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We first note that: $\mathcal{L}(x_i) = \mathcal{L}_i(x_i)$ for $i = 0, \ldots, n-1$ and $\mathcal{L}(x_n) = \mathcal{L}_{n-1}(x_n)$. Hence by using (C.1) we obtain:

$$\widehat{g}_i - \gamma\tau \le \mathcal{L}_{(}x_i) \le \widehat{g}_i + \gamma\tau \Leftrightarrow \widehat{g}_i - \gamma\tau \le b_i h + \frac{a_i h^2}{6} \le \widehat{g}_i + \gamma\tau, \ i = 0, \ldots, n-1,$$
(C.21)

$$\widehat{g}_n - \gamma\tau \le \mathcal{L}_{n-1}(x_n) \le \widehat{g}_n + \gamma\tau \Leftrightarrow \widehat{g}_n - \gamma\tau \le c_{n-1} h + \frac{a_n h^2}{6} \le \widehat{g}_n + \gamma\tau.$$
(C.22)

2. **Continuity of $\mathcal{L}, \mathcal{L}', \mathcal{L}''$ at $\mathbf{x_1}, \ldots, \mathbf{x_{n-1}}$.**

   First note that:

$$\mathcal{L}(x_i^-) = \mathcal{L}(x_i^+); \quad i = 1, \ldots, n-1 \tag{C.23}$$
$$\Leftrightarrow \mathcal{L}_i(x_{i+1}) = \mathcal{L}_{i+1}(x_{i+1}); \quad i = 0, \ldots, n-2 \tag{C.24}$$
$$\Leftrightarrow a_{i+1}\frac{h^2}{6} + c_i h = a_{i+1}\frac{h^2}{6} + b_{i+1} h; \quad i = 0, \ldots, n-2 \tag{C.25}$$
$$\Leftrightarrow c_i h = b_{i+1} h; \quad i = 0, \ldots, n-2. \tag{C.26}$$

   Next note that:

$$\mathcal{L}'(x_i^-) = \mathcal{L}'(x_i^+); \quad i = 1, \ldots, n-1 \tag{C.27}$$
$$\Leftrightarrow \mathcal{L}'_i(x_{i+1}) = \mathcal{L}'_{i+1}(x_{i+1}); \quad i = 0, \ldots, n-2 \tag{C.28}$$
$$\Leftrightarrow a_{i+1}\frac{h}{2} - b_i + c_i = -a_{i+1}\frac{h}{2} - b_{i+1} + c_{i+1}; \quad i = 0, \ldots, n-2 \tag{C.29}$$
$$\Leftrightarrow a_{i+1} h = (b_i - c_i) + (c_{i+1} - b_{i+1}); \quad i = 0, \ldots, n-2. \tag{C.30}$$

   The continuity of $\mathcal{L}''$ at $x_1, \ldots, x_{n-1}$ is already ensured through the choice of a_i's.

3. **Continuity of $\mathcal{L}'''$ at $\mathbf{x_1}, \mathbf{x_{n-1}}$ (not-a-knot boundary conditions).**

   It is easily verifiable that $\mathcal{L}'''_i(x) = -\frac{a_i}{h} + \frac{a_{i+1}}{h}$. Hence we have that

$$\mathcal{L}'''_0(x_1) = \mathcal{L}'''_1(x_1) \Leftrightarrow a_0 - 2a_1 + a_2 = 0, \tag{C.31}$$
$$\mathcal{L}'''_{n-2}(x_{n-1}) = \mathcal{L}'''_{n-1}(x_{n-1}) \Leftrightarrow a_{n-2} - 2a_{n-1} + a_n = 0. \tag{C.32}$$

$\square$

C.3 Putting it together: Proof of Theorem 2

We prove a more general form of Theorem 2 namely the following.

Theorem 6. *For $g \in C^4[t_1, t_2]$ let $\mathcal{L}^* : [t_1, t_2] \to \mathbb{R}$ be a solution of (P) for some parameter $\gamma \ge 0$.*

1. *(General bound) For any $\gamma \ge 0$ we have that*

$$\| \mathcal{L}^* - g \|_\infty \le \left[\frac{118(1+\gamma)}{3} \right] \tau + \frac{29}{64} h^4 \| g^{(4)} \|_\infty. \tag{C.33}$$

2. *(Bound under large perturbation) For any $\gamma > 1$, if*

$$\tau \ge \frac{14h^2\lambda}{(\gamma-1)} \left(\| g'' \|_\infty + \frac{h^2}{12} \| g^{(4)} \|_\infty \right) \tag{C.34}$$

is satisfied where $\lambda = \Theta(n^{5/2})$ then $\mathcal{L}^*$ is a line and we have that

$$\| \mathcal{L}^* - g \|_\infty \le 2\tau \left[(\gamma+1) + \frac{\gamma-1}{3\lambda} \right] + \frac{29}{64} h^4 \| g^{(4)} \|_\infty. \tag{C.35}$$

Proof. **General error bound.** Let $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as defined in (C.2) be a feasible point satisfying (C.11)-(C.15). From (C.11),(C.12) we have that any feasible point satisfies:

$$b_i h = \widehat{g}_i - \frac{a_i h^2}{6} + e_i; \; i = 0, \ldots, n-1 \tag{C.36}$$

$$c_{n-1} h = \widehat{g}_n - \frac{a_n h^2}{6} + e_n, \tag{C.37}$$

for some e_i with $|e_i| < \gamma\tau$. Plugging (C.36),(C.37), (C.13) in (C.14) we obtain for $i = 0, \ldots, n-2$:

$$a_{i+1} h = \frac{\widehat{g}_i - \frac{a_i h^2}{6} + e_i}{h} - 2\frac{\widehat{g}_{i+1} - \frac{a_{i+1} h^2}{6} + e_{i+1}}{h} + \frac{\widehat{g}_{i+2} - \frac{a_{i+2} h^2}{6} + e_{i+2}}{h}, \tag{C.38}$$

$$\Leftrightarrow a_{i+1} h = \left(\frac{\widehat{g}_i - 2\widehat{g}_{i+1} + \widehat{g}_{i+2}}{h} \right) + \left(\frac{a_{i+1} h}{3} - \frac{a_i h}{6} - \frac{a_{i+2} h}{6} \right) + \left(\frac{e_i - 2e_{i+1} + e_{i+2}}{h} \right), \tag{C.39}$$

$$\Leftrightarrow \frac{2a_{i+1} h}{3} + \frac{a_i h}{6} + \frac{a_{i+2} h}{6} = \left(\frac{\widehat{g}_i - 2\widehat{g}_{i+1} + \widehat{g}_{i+2}}{h} \right) + \left(\frac{e_i - 2e_{i+1} + e_{i+2}}{h} \right), \tag{C.40}$$

$$\Leftrightarrow a_i + 4_{i+1} + a_{i+2} = \frac{6}{h^2}(\widehat{g}_i - 2\widehat{g}_{i+1} + \widehat{g}_{i+2}) + \frac{6}{h^2}(e_i - 2e_{i+1} + e_{i+2}). \tag{C.41}$$

Now since $|e_i| \leq \gamma\tau$ we have that $\frac{6}{h^2} |(e_i - 2e_{i+1} + e_{i+2})| \leq \frac{24\gamma\tau}{h^2}$ for $i = 0, \ldots, n-2$. Furthermore, since $|\widehat{g}_i - g_i| \leq \tau$, for $i = 0, \ldots, n-2$ we have that

$$a_i + 4_{i+1} + a_{i+2} = \frac{6}{h^2}(g_i - 2g_{i+1} + g_{i+2}) + \eta_i; \quad i = 0, \ldots, n-2 \tag{C.42}$$

where $|\eta_i| \leq \frac{24(1+\gamma)\tau}{h^2}$. Thus (C.42) together with the boundary conditions (C.15) give us the following linear system of equations, perturbed by the noise vector η:

$$\underbrace{\begin{pmatrix} 1 & -2 & 1 & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & 1 & -2 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}}_{\mathbf{a}} = \frac{6}{h^2} \underbrace{\begin{pmatrix} 0 \\ g_0 - 2g_1 + g_2 \\ \vdots \\ \vdots \\ g_{n-2} - 2g_{n-1} + g_n \\ 0 \end{pmatrix}}_{\mathbf{g}} + \underbrace{\begin{pmatrix} 0 \\ \eta_0 \\ \vdots \\ \vdots \\ \eta_{n-2} \\ 0 \end{pmatrix}}_{\eta} \tag{C.43}$$

Now from Proposition 4 we know that $\| \mathbf{A}^{-1} \|_\infty \leq 7/3$. Also note that

$$\mathbf{a}_{or} := \mathbf{A}^{-1}\left(\frac{6}{h^2} \mathbf{g} \right) = (a_{0,or} \; a_{1,or} \; \cdots a_{n,or}) \tag{C.44}$$

denotes the (unique) coefficient vector that would have been obtained as solution in the noiseless setting. From (C.43) we then have that:

$$\mathbf{a} = \underbrace{\mathbf{A}^{-1}\left(\frac{6}{h^2} \mathbf{g} \right)}_{\mathbf{a}_{or}} + \mathbf{A}^{-1} \eta \tag{C.45}$$

$$\Rightarrow \| \mathbf{a} - \mathbf{a}_{or} \|_\infty \leq \| \mathbf{A}^{-1} \|_\infty \| \eta \|_\infty \leq \frac{56(1+\gamma)\tau}{h^2}. \tag{C.46}$$

Let $\mathbf{b}_{or} := (b_{0,or} \; b_{1,or} \; \cdots b_{n-1,or})$ and $\mathbf{c}_{or} := (c_{0,or} \; c_{1,or} \; \cdots c_{n-1,or})$ denote the optimal spline coefficients in the noise less setting ($\tau = 0$). Then, $b_{i,or} h = g_i - a_{i,or}\frac{h^2}{6}$; $i = 0, \ldots, n-1$. Furthermore $c_{i,or} = b_{i+1,or}$; $i = 0, \ldots, n-2$ and $c_{n-1,or} = g_n - a_{n,or}\frac{h^2}{6}$.

Now recall from (C.36) that $b_i h = \widehat{g}_i - a_i \frac{h^2}{6} + e_i$ where $|e_i| < \gamma\tau$. Since $b_{i,or} h = g_i - a_{i,or} \frac{h^2}{6}$, this then implies that

$$\| \mathbf{b} - \mathbf{b}_{or} \|_\infty = \max_{0 \le i \le n-1} |b_i - b_{i,or}|, \tag{C.47}$$

$$\le \max_{0 \le i \le n-1} \frac{1}{h} \left(|\widehat{g}_i - g_i| + |e_i| + \frac{h^2}{6} |a_{i,or} - a_i| \right), \tag{C.48}$$

$$\le \frac{1}{h} \left((\gamma + 1)\tau + \frac{28(1+\gamma)\tau}{3} \right) = \frac{31(1+\gamma)\tau}{3h}. \tag{C.49}$$

Analogously, it can be verified that $\| \mathbf{c} - \mathbf{c}_{or} \|_\infty \le \frac{31(1+\gamma)\tau}{3h}$.

Now let $\mathcal{L}_{or}$ denote the (unique) cubic spline obtained in the noiseless setting ($\tau = 0$) so that $\mathcal{L}_{or}(x) = \mathcal{L}_{i,or}(x)$ for $x \in [x_i, x_{i+1}]$ where

$$\mathcal{L}_{or}(x) = \frac{a_{i,or}(x_{i+1} - x)^3}{6h} + \frac{a_{i+1,or}(x - x_i)^3}{6h} + b_{i,or}(x_{i+1} - x) + c_{i,or}(x - x_i). \tag{C.50}$$

Let $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ be a solution to (P) and $\mathcal{L}^*$ denote the corresponding cubic spline. We then have for $x \in [x_i, x_{i+1}]$ that :

$$|\mathcal{L}_i^*(x) - \mathcal{L}_{i,or}(x)| = |(a_i^* - a_{i,or})\frac{(x_{i+1} - x)^3}{6h} + (a_{i+1}^* - a_{i+1,or})\frac{(x - x_i)^3}{6h} \tag{C.51}$$

$$+ (b_i^* - b_{i,or})(x_{i+1} - x) + (c_i^* - c_{i,or})(x - x_i)|, \tag{C.52}$$

$$\le \| \mathbf{a}^* - \mathbf{a}_{or} \|_\infty \left| \frac{(x_{i+1} - x)^3}{6h} \right| + \| \mathbf{a}^* - \mathbf{a}_{or} \|_\infty \left| \frac{(x - x_i)^3}{6h} \right| \tag{C.53}$$

$$+ \| \mathbf{b}^* - \mathbf{b}_{or} \|_\infty |x_{i+1} - x| + \| \mathbf{c}^* - \mathbf{c}_{or} \|_\infty |x - x_i|, \tag{C.54}$$

$$\le \frac{56(1+\gamma)\tau}{3} + \frac{62(1+\gamma)\tau}{3} = \frac{118(1+\gamma)\tau}{3}. \tag{C.55}$$

This then implies that

$$\| \mathcal{L}^* - \mathcal{L}_{or} \|_\infty = \max_{0 \le i \le n-1} \| \mathcal{L}_i^* - \mathcal{L}_{i,or} \|_\infty \le \frac{118(1+\gamma)\tau}{3}. \tag{C.56}$$

Since $\| \mathcal{L}_{or} - g \|_\infty \le \frac{29}{64} h^4 \| g^{(4)} \|_\infty$ by Proposition 6, we have by triangles inequality the final error bound:

$$\| \mathcal{L}^* - g \|_\infty \le \| \mathcal{L}^* - \mathcal{L}_{or} \|_\infty + \| \mathcal{L}_{or} - g \|_\infty, \tag{C.57}$$

$$\le \frac{118(1+\gamma)\tau}{3} + \frac{29}{64} h^4 \| g^{(4)} \|_\infty. \tag{C.58}$$

Bound under large perturbation. Let $\mathbf{a}_{or}, \mathbf{b}_{or}, \mathbf{c}_{or}$ denote the unique solution when $\tau = 0$, i.e. in the noiseless setting. Recall that $\mathbf{a}_{or}$ has the form:

$$\mathbf{a}_{or} = \mathbf{A}^{-1} \left(\frac{6}{h^2} \right) \begin{pmatrix} 0 \\ g_0 - 2g_1 + g_2 \\ \vdots \\ \vdots \\ g_{n-2} - 2g_{n-1} + g_n \\ 0 \end{pmatrix}. \tag{C.59}$$

Using Proposition 3(1) we then obtain:

$$\| \mathbf{a}_{or} \|_\infty \le 6 \| \mathbf{A}^{-1} \|_\infty \max_{1 \le i \le n-1} \left(\left| g_i'' + \frac{h^2}{12} \| g^{(4)} \|_\infty \right| \right),$$

$$\le 14 \left(\| g'' \|_\infty + \frac{h^2}{12} \| g^{(4)} \|_\infty \right), \tag{C.60}$$

where we used $\| \mathbf{A}^{-1} \|_\infty \leq 7/3$ from Proposition 4. Recall also that:

$$b_{i,or} = \frac{g_i}{h} - \frac{a_{i,or}h}{6}; \quad i = 0, \ldots, n-1, \tag{C.61}$$

$$c_{i,or} = b_{i+1,or}; \quad i = 0, \ldots, n-2, \tag{C.62}$$

$$\text{and} \quad c_{n-1,or} = \frac{g_n}{h} - \frac{a_{n,or}h}{6}. \tag{C.63}$$

Let $(\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*)$ be a solution to (P). Then $\mathbf{a}^* = \mathbf{0}$ iff there exists feasible $\mathbf{b}^*, \mathbf{c}^*$ satisfying:

$$b_i^* \in \left[\frac{\widehat{g}_i - \gamma\tau}{h}, \frac{\widehat{g}_i + \gamma\tau}{h} \right]; \quad i = 0, \ldots, n-1, \tag{C.64}$$

$$\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \begin{pmatrix} b_1^* \\ b_2^* \\ \vdots \\ \vdots \\ b_{n-2}^* \\ b_{n-1}^* \end{pmatrix} = \begin{pmatrix} -b_0^* \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -c_{n-1}^* \end{pmatrix}, \tag{C.65}$$

$$c_{n-1}^* \in \left[\frac{\widehat{g}_n - \gamma\tau}{h}, \frac{\widehat{g}_n + \gamma\tau}{h} \right]. \tag{C.66}$$

Here (C.65) follows by plugging (C.13) in (C.14) and using $\mathbf{a}^* = \mathbf{0}$. Now we know that $\mathbf{b}_{or}, \mathbf{c}_{or}$ satisfy:

$$\underbrace{\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix}}_{\mathbf{B}} \begin{pmatrix} b_{1,or} \\ b_{2,or} \\ \vdots \\ \vdots \\ b_{n-2,or} \\ b_{n-1,or} \end{pmatrix} = \begin{pmatrix} -b_{0,or} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -c_{n-1,or} \end{pmatrix} + h \begin{pmatrix} a_{1,or} \\ a_{2,or} \\ \vdots \\ \vdots \\ a_{n-2,or} \\ a_{n-1,or} \end{pmatrix}. \tag{C.67}$$

This follows simply from the constraints (C.13),(C.14). Since from Proposition 5 we have that $\mathbf{B}$ is invertible, hence we can rewrite (C.67) to obtain equivalently:

$$\mathbf{B} \left[\begin{pmatrix} b_{1,or} \\ b_{2,or} \\ \vdots \\ \vdots \\ b_{n-2,or} \\ b_{n-1,or} \end{pmatrix} - h\mathbf{B}^{-1} \underbrace{\begin{pmatrix} a_{1,or} \\ a_{2,or} \\ \vdots \\ \vdots \\ a_{n-2,or} \\ a_{n-1,or} \end{pmatrix}}_{\mathbf{a}'} \right] = \begin{pmatrix} -b_{0,or} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -c_{n-1,or} \end{pmatrix}. \tag{C.68}$$

Observe that (C.68) holds for $\mathbf{a}_{or}, \mathbf{b}_{or}, \mathbf{c}_{or}$. Thus if we can find conditions on τ such that

$$\frac{\widehat{g}_0 - \gamma\tau}{h} \leq \underbrace{\frac{g_0}{h} - \frac{a_{0,or}h}{6}}_{b_{0,or}} \leq \frac{\widehat{g}_0 + \gamma\tau}{h}, \tag{C.69}$$

$$\frac{\widehat{g}_i - \gamma\tau}{h} \leq \underbrace{\left(\frac{g_i}{h} - \frac{a_{i,or}h}{6} \right)}_{b_{i,or}} - h(\mathbf{B}^{-1}\mathbf{a}')_i \leq \frac{\widehat{g}_i + \gamma\tau}{h}; \quad i = 1, \ldots, n-1, \tag{C.70}$$

$$\frac{\widehat{g}_n - \gamma\tau}{h} \leq \underbrace{\frac{g_n}{h} - \frac{a_{n,or}h}{6}}_{c_{n-1,or}} \leq \frac{\widehat{g}_n + \gamma\tau}{h} \tag{C.71}$$

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are satisfied then we are done. Let us look at (C.70) first. We equivalently obtain for each $i = 1, \ldots, n-1$:

$$\frac{\gamma\tau}{h} \geq \frac{g_i - \widehat{g}_i}{h} - \frac{a_{i,or}h}{6} - h(\mathbf{B}^{-1}\mathbf{a}')_i, \tag{C.72}$$

$$\frac{\gamma\tau}{h} \geq \frac{\widehat{g}_i - g_i}{h} + \frac{a_{i,or}h}{6} + h(\mathbf{B}^{-1}\mathbf{a}')_i. \tag{C.73}$$

Clearly, R.H.S of (C.72),(C.73) is less than or equal to: $\frac{\tau}{h} + \frac{\|\mathbf{a}_{or}\|_\infty h}{6} + h \parallel \mathbf{B}^{-1} \parallel_\infty \parallel \mathbf{a}_{or} \parallel_\infty$. Thus a sufficient condition for (C.72),(C.73) to hold is:

$$\frac{\gamma\tau}{h} \geq \frac{\tau}{h} + \frac{\parallel \mathbf{a}_{or} \parallel_\infty h}{6} + h \parallel \mathbf{B}^{-1} \parallel_\infty \parallel \mathbf{a}_{or} \parallel_\infty, \tag{C.74}$$

$$\Leftrightarrow \tau \geq \frac{h^2}{\gamma - 1} \parallel \mathbf{a}_{or} \parallel_\infty \left(\frac{1}{6} + \parallel \mathbf{B}^{-1} \parallel_\infty \right). \tag{C.75}$$

Using (C.60) and Proposition 5 the sufficient condition becomes:

$$\tau \geq \frac{14h^2}{\gamma - 1} \left(\parallel g'' \parallel_\infty + \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty \right) \left(\frac{1}{6} + \frac{n^2}{\pi^2} \sqrt{n-1} \left(1 - \frac{\pi^2}{12} \right)^{-1} \right). \tag{C.76}$$

Observe that the above condition on τ also guarantees that (C.69), (C.71) are satisfied. Hence if the error τ is such that it satisfies (C.76) we then have that $\mathbf{a}^* = 0$. Using (C.60), this then implies that:

$$\parallel \mathbf{a}_{or} - \mathbf{a}^* \parallel_\infty \leq 14 \left(\parallel g'' \parallel_\infty + \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty \right). \tag{C.77}$$

Furthermore since

$$b_{i,or} = \frac{g_i}{h} - \frac{a_{i,or}h}{6}, \text{ and } b_i^* \in \left[\frac{\widehat{g}_i - \gamma\tau}{h}, \frac{\widehat{g}_i + \gamma\tau}{h} \right]; \quad i = 0, \ldots, n-1 \tag{C.78}$$

holds, we have that:

$$\parallel \mathbf{b}_{or} - \mathbf{b}^* \parallel_\infty \leq \frac{\tau}{h} + \frac{\gamma\tau}{h} + \parallel \mathbf{a}_{or} \parallel_\infty \frac{h}{6}, \tag{C.79}$$

$$\leq \frac{\tau(\gamma + 1)}{h} + \frac{7h}{3} \left(\parallel g'' \parallel_\infty + \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty \right). \tag{C.80}$$

Analogously we obtain

$$\parallel \mathbf{c}_{or} - \mathbf{c}^* \parallel_\infty \leq \frac{\tau(\gamma + 1)}{h} + \frac{7h}{3} \left(\parallel g'' \parallel_\infty + \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty \right). \tag{C.81}$$

Let $\mathcal{L}_{or}, \mathcal{L}^*$ be the cubic splines corresponding to $(\mathbf{a}_{or}, \mathbf{b}_{or}, \mathbf{c}_{or})$ and $(\mathbf{a}^*(= 0), \mathbf{b}^*, \mathbf{c}^*)$. We then have for $x \in [x_i, x_{i+1}]$:

$$\begin{aligned}
|\mathcal{L}_i^*(x) - \mathcal{L}_{i,or}(x)| &= \left| (a_i^* - a_{i,or})\frac{(x_{i+1} - x)^3}{6h} + (a_{i+1}^* - a_{i+1,or})\frac{(x - x_i)^3}{6h} \right. & \text{(C.82)} \\
&+ (b_i^* - b_{i,or})(x_{i+1} - x) + (c_i^* - c_{i,or})(x - x_i)|, & \text{(C.83)} \\
&\leq \parallel \mathbf{a}^* - \mathbf{a}_{or} \parallel_\infty \left| \frac{(x_{i+1} - x)^3}{6h} \right| + \parallel \mathbf{a}^* - \mathbf{a}_{or} \parallel_\infty \left| \frac{(x - x_i)^3}{6h} \right| & \text{(C.84)} \\
&+ \parallel \mathbf{b}^* - \mathbf{b}_{or} \parallel_\infty |x_{i+1} - x| + \parallel \mathbf{c}^* - \mathbf{c}_{or} \parallel_\infty |x - x_i|, & \text{(C.85)} \\
&\leq \parallel \mathbf{a}^* - \mathbf{a}_{or} \parallel_\infty \frac{h^2}{3} + 2h \parallel \mathbf{b}^* - \mathbf{b}_{or} \parallel_\infty, & \text{(C.86)} \\
&\leq 2(\gamma + 1)\tau + \frac{28h^2}{3} \left(\parallel g'' \parallel_\infty + \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty \right). & \text{(C.87)}
\end{aligned}$$

This in turn implies that

$$\parallel \mathcal{L}^* - \mathcal{L}_{or} \parallel_\infty = \max_{0 \leq i \leq n-1} \parallel \mathcal{L}_i^* - \mathcal{L}_{i,or} \parallel_\infty \leq 2(\gamma + 1)\tau + \frac{28h^2}{3} \left(\parallel g'' \parallel_\infty + \frac{h^2}{12} \parallel g^{(4)} \parallel_\infty \right). \tag{C.88}$$

Since by Proposition 6: $\parallel \mathcal{L}_{or} - g \parallel_\infty \leq \frac{29}{64}h^4 \parallel g^{(4)} \parallel_\infty$, we finally obtain the stated error bound by employing (C.76) in (C.88), and then by using triangles inequality. $\square$

C.4 Uniqueness of the solution of (P)

In this section we will discuss about the uniqueness of the solution returned by the convex program (P). We state this in the form of the following theorem.

Theorem 7. *Let* $\mathcal{L}^* : [t_1, t_2] \to \mathbb{R}$ *be one of the optimal solutions to (P), for any fixed* $\gamma \geq 0$. *If* $\int_{t_1}^{t_2} (\mathcal{L}^{*\prime\prime}(x))^2 dx > 0$, *then* $\mathcal{L}^*$ *is a unique solution.*

Proof. Let $\mathbf{a}^* = (a_0^*, \ldots, a_n^*)$, $\mathbf{b}^* = (b_0^*, \ldots, b_{n-1}^*)$, $\mathbf{c}^* = (c_0^*, \ldots, c_{n-1}^*)$ be a solution to (P), that corresponds to the spline $\mathcal{L}^*$. This implies that $\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*$ satisfy:

$$\widehat{g}_i - \gamma\tau \leq b_i h + \frac{a_i h^2}{6} \leq \widehat{g}_i + \gamma\tau; \quad i = 0, \ldots, n-1, \tag{C.89}$$

$$\widehat{g}_n - \gamma\tau \leq c_{n-1} h + \frac{a_n h^2}{6} \leq \widehat{g}_n + \gamma\tau, \tag{C.90}$$

$$\underbrace{\begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}}_{\mathbf{B}} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} -b_0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ -c_{n-1} \end{pmatrix} + h \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_{n-2} \\ a_{n-1} \end{pmatrix}, \tag{C.91}$$

$$a_0 - 2a_1 + a_2 = 0 \quad \text{and} \quad a_{n-2} - 2a_{n-1} + a_n = 0. \tag{C.92}$$

Here, (C.91) is derived from (C.13), (C.14) leading to the variables $c_0, \ldots, c_{n-2}$ being eliminated. We also have that $\int_{t_1}^{t_2} (\mathcal{L}^{*\prime\prime}(x))^2 dx = \frac{h}{6} \mathbf{a}^{*T} Q \mathbf{a}^*$, where Q is the (positive definite) matrix in (C.10). Therefore, since $\int_{t_1}^{t_2} (\mathcal{L}^{*\prime\prime}(x))^2 dx > 0$ we must have at least one non-zero a_i^*.

Let us denote the ith constraint in (C.89) by E_i and the constraint in (C.90) by E_n.

Claim 1. *Not all the* E_i; $i = 0, \ldots, n$ *are satisfied strictly by* $\mathbf{a}^*, \mathbf{b}^*, c_{n-1}^*$. *In other words we cannot have that*

$$\widehat{g}_i - \gamma\tau < b_i^* h + \frac{a_i^* h^2}{6} < \widehat{g}_i + \gamma\tau; \quad i = 0, \ldots, n-1,$$

$$\widehat{g}_n - \gamma\tau < c_{n-1}^* h + \frac{a_n^* h^2}{6} < \widehat{g}_n + \gamma\tau.$$

Proof. Let us assume the contrary. Then this implies that $\exists t \in (0, 1)$, such that $(t\mathbf{a}^*, t\mathbf{b}^*, tc_{n-1}^*)$ is also feasible. However $\int_{t_1}^{t_2} (\mathcal{L}^{*\prime\prime}(x))^2 dx$ has a smaller value at $t\mathbf{a}^*$ than $\mathbf{a}^*$, which is a contradiction to $\mathbf{a}^*$ being optimal. $\qquad\square$

Now say that $\mathbf{a}' = (a_0', \ldots, a_n')$, $\mathbf{b}' = (b_0', \ldots, b_{n-1}')$, c_{n-1}' be another solution to (P) where: $\mathbf{a}' \neq \mathbf{a}^*$ and $\mathbf{a}^{*T} Q \mathbf{a}^* = \mathbf{a}'^{T} Q \mathbf{a}'$. Clearly, any $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{c}_{n-1}$ with: $\tilde{\mathbf{a}} = \lambda \mathbf{a}^* + (1-\lambda)\mathbf{a}'$, $\tilde{\mathbf{b}} = \lambda \mathbf{b}^* + (1-\lambda)\mathbf{b}'$, $\tilde{c}_{n-1} = \lambda c_{n-1}^* + (1-\lambda)c_{n-1}'$ for $\lambda \in (0, 1)$ is also feasible. However, it can be verified that for $\lambda \in (0, 1)$, we have $\tilde{\mathbf{a}}^T Q \tilde{\mathbf{a}} < \mathbf{a}^{*T} Q \mathbf{a}^*$ which is a contradiction.

Let $\mathbf{a}^*, \mathbf{b}' = (b_0', \ldots, b_{n-1}'), c_{n-1}'$ be another solution to (P), corresponding to a spline $\mathcal{L}_1$, with $\mathcal{L}_1 \neq \mathcal{L}^*$. This means that *at least* one of b_i^*, c_{n-1}^* differs from b_i', c_{n-1}'; $i = 0, \ldots, n-1$.

Claim 2. *It is necessary that:* $b_0^* \neq b_0'$ *and/or* $c_{n-1}^* \neq c_{n-1}'$

Proof. In case both $b_0^* = b_0'$ and $c_{n-1}^* = c_{n-1}'$ holds, then due to invertibility of $\mathbf{B}$, we have $b_i^* = b_i'$; $i = 0, \ldots, n-1$. This means that $\mathcal{L}_1 = \mathcal{L}^*$, contradicting our assumption that they are different. $\qquad\square$

Lemma 6. *If* $b_0^* \neq b_0'$ *and/or* $c_{n-1}^* \neq c_{n-1}'$, *then it implies that the solution* $\mathbf{a}^*, \mathbf{b}^*, c_{n-1}^*$ *is unique.*

Proof. On account of Claim 2, we first consider the scenario where $b'_0 = b^*_0 + \epsilon$ for some $\epsilon > 0$. For $\epsilon_1, \epsilon_2 \geq 0$ we then have one of the following four possibilities for $\mathbf{b}' = (b'_0, \ldots, b'_{n-1}), c'_{n-1}$.

1. **Case 1.** $b'_1 = b^*_1 + \epsilon_1$ and $b'_2 = b^*_2 + \epsilon_2$. Due to (C.91), this then implies that $2\epsilon_1 - \epsilon_2 = \epsilon$ and

$$b'_i = b^*_i - [(i-2)\epsilon_1 - (i-1)\epsilon_2]; \quad i = 3, \ldots, n-1,$$
$$c'_{n-1} = c^*_{n-1} - [(n-2)\epsilon_1 - (n-1)\epsilon_2].$$

2. **Case 2.** $b'_1 = b^*_1 - \epsilon_1$ and $b'_2 = b^*_2 + \epsilon_2$. Due to (C.91), this then implies that $2\epsilon_1 + \epsilon_2 = -\epsilon$ which is impossible. Hence this case cannot occur.

3. **Case 3.** $b'_1 = b^*_1 + \epsilon_1$ and $b'_2 = b^*_2 - \epsilon_2$, which on account of (C.91) implies that $2\epsilon_1 + \epsilon_2 = \epsilon$ and

$$b'_i = b^*_i - [(i-2)\epsilon_1 + (i-1)\epsilon_2]; \quad i = 3, \ldots, n-1,$$
$$c'_{n-1} = c^*_{n-1} - [(n-2)\epsilon_1 + (n-1)\epsilon_2].$$

4. **Case 4.** $b'_1 = b^*_1 - \epsilon_1$ and $b'_2 = b^*_2 - \epsilon_2$. On account of (C.91), this then implies that $\epsilon_2 - 2\epsilon_1 = \epsilon$ and

$$b'_i = b^*_i - [(i-1)\epsilon_2 - (i-2)\epsilon_1]; \quad i = 3, \ldots, n-1,$$
$$c'_{n-1} = c^*_{n-1} - [(n-1)\epsilon_2 - (n-2)\epsilon_1].$$

Say that it is possible to obtain $\mathbf{b}', c'_{n-1}$ in one of the above cases, for some value of $\epsilon > 0$ and $\epsilon_1, \epsilon_2 \geq 0$. We then make the following claim.

Claim 3. *There will always exist $\epsilon', \epsilon'_1, \epsilon'_2 > 0$ resulting in $\mathbf{b}', c'_{n-1}$ such that each $E_i; i = 0, \ldots, n$ is satisfied strictly by $\mathbf{a}^*, \mathbf{b}', c'_{n-1}$.*

Proof. Consider Case 1 first. Denote $P = \left\{ 0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n-2}{n-1} \right\}$. We necessarily have $\epsilon_1 > 0$, however $\epsilon_2 \geq 0$. Let $\epsilon_2 = \ell\epsilon_1$, where $0 \leq \ell < 2$. This implies that

$$\epsilon_1 = \frac{\epsilon}{2 - \ell}, \quad \epsilon_2 = \frac{\ell\epsilon}{2 - \ell},$$
$$b'_i = b^*_i - \frac{\epsilon}{2 - \ell}[(i-2) - (i-1)\ell]; \quad i = 3, \ldots, n-1,$$
$$c'_{n-1} = c^*_{n-1} - \frac{\epsilon}{2 - \ell}[(n-2) - (n-1)\ell].$$

If $\ell \in P$ then at most one b^*_i or c^*_{n-1} is equal to the corresponding b'_i or c'_{n-1}, repsectively. For any other ℓ, we will have that $b^*_i \neq b'_i$ and $c^*_{n-1} \neq c'_{n-1}$; $i = 0, \ldots, n-1$. Note that if $\ell \notin P$, then $\exists \epsilon' < \epsilon$ such that with ℓ, ϵ', we have that each b^*_i and c^*_{n-1} changes by a non-zero amount with b'_i, c'_{n-1} lying strictly inside the intervals defined in (C.89),(C.90). In case $\ell \in P$, we have for $\ell' = \beta + \ell$, with $\beta \neq 0$ chosen appropriately, that $\ell' \notin P$. For this ℓ' we can then choose $\epsilon' < \epsilon$ suitably to ensure that each b^*_i and c^*_{n-1} changes by a non-zero amount with b'_i, c'_{n-1} strictly satisfying (C.89),(C.90). Similar arguments can be made for Case 3 and Case 4. $\square$

Consequently, with the help of Claim 1, we arrive at a contradiction regarding optimality of $\mathbf{a}^*$. This implies that none of Cases $1 - 4$ are possible. In the scenario where $b'_0 = b^*_0 - \epsilon$ for some $\epsilon > 0$, we arrive via an analogous case analysis at the same conclusion as above.

Lastly, we could have also started the above analysis by considering $c'_{n-1} = c^*_{n-1} \pm \epsilon$, for $\epsilon > 0$. Indeed, by considering different possibilities for perturbation of b^*_{n-1}, b^*_{n-2}, we will obtain perturbation expressions for all: $b'_{n-3}, b'_{n-4}, \ldots, b'_1, b'_0$. By an analogous case analysis as shown above, we can again show that none of the cases are possible. This means that $\nexists \mathbf{a}^*, \mathbf{b}', c'_{n-1}$ different from $\mathbf{a}^*, \mathbf{b}^*, c^*_{n-1}$, as a solution to (P).

$\square$

$\square$

D Proof of Theorem 1

In this section, we will prove a general version of Theorem 1 namely the following.

Theorem 8. *There exist constants $C, C_1 > 0$ such that if $m_x \geq (1/\delta)$, $m_v \geq C_1 k \log d$, $0 < \epsilon < \frac{D\sqrt{m_v}}{CkB_2}$ and $\tau = \frac{C\epsilon k B_2}{2\sqrt{m_v}}$ then with high probability, $\widehat{S} = S$ and for any $\gamma \geq 0$ the estimate $\phi_{est,l}$ returned by Algorithm 1 satisfies for each $l \in S$:*

$$\| \phi_{est,l} - \phi_l \|_{L^\infty[-1,1]} \leq [59(1+\gamma)] \frac{C\epsilon k B_2}{\sqrt{m_v}} + \frac{87}{64 m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} . \tag{D.1}$$

Furthermore for any $\gamma > 1$ suppose that ϵ additionally satisfies

$$\epsilon > C_3 \sqrt{m_v m_x} \left(\| \phi_l''' \|_{L^\infty[-1,1]} + \frac{1}{12 m_x^2} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} \right) \tag{D.2}$$

for some constant C_3 that depends on C, k, B_2, γ. Then the estimate $\phi_{est,l}$ is a polynomial of degree at most 2 and:

$$\| \phi_{est,l} - \phi_l \|_{L^\infty[-1,1]} \leq \left[3(\gamma+1) + \frac{\gamma-1}{\lambda} \right] \frac{C\epsilon k B_2}{\sqrt{m_v}} + \frac{87}{64 m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} . \tag{D.3}$$

To this end we first have the following Corollary of Theorem 6 for estimation of C^4 smooth ϕ_l' in the interval $[-1, 1]$. Corollary 3 can be seen as a generalized version of Corollary 1.

Corollary 3. *Let (P) be employed for each $l \in S$ using noisy samples $\left\{ \widehat{\phi}'_l(i/m_x) \right\}_{i=-m_x}^{m_x}$, and with step size ϵ satisfying $0 < \epsilon < \frac{D\sqrt{m_v}}{CkB_2}$. Denoting $\tilde{\phi}'_l$ as the corresponding solution returned by (P), we then have the following results.*

1. *(**General bound**) For any $\gamma \geq 0$:*

$$\| \tilde{\phi}'_l - \phi_l' \|_{L^\infty[-1,1]} \leq \left[\frac{59(1+\gamma)}{3} \right] \frac{C\epsilon k B_2}{\sqrt{m_v}} + \frac{29}{64 m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} . \tag{D.4}$$

2. *(**Large step size** ϵ) For any $\gamma > 1$ if ϵ additionally satisfies*

$$\epsilon > \frac{28\lambda_1}{CkB_2(\gamma-1)} \left(\| \phi_l''' \|_{L^\infty[-1,1]} + \frac{1}{12 m_x^2} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} \right) \tag{D.5}$$

where $\lambda_1 = \Theta(\sqrt{m_x m_v})$ we then have that $\tilde{\phi}'_l$ is a line and:

$$\| \tilde{\phi}'_l - \phi_l' \|_{L^\infty[-1,1]} \leq \left[(\gamma+1) + \frac{\gamma-1}{3\lambda} \right] \frac{C\epsilon k B_2}{\sqrt{m_v}} + \frac{29}{64 m_x^4} \| \phi_l^{(5)} \|_{L^\infty[-1,1]} . \tag{D.6}$$

The proof simply involves replacing: g with ϕ_l', $n+1$ with $2m_x+1$, h with $1/m_x$ and τ with $\frac{C\epsilon k B_2}{2\sqrt{m_v}}$. As the perturbation τ is directly proportional to the step size ϵ, hence the large perturbation scenario in Theorem 6 translates to a lower bound on ϵ.

Note that Lemma 1 together with Corollary 3 almost completes the proof of Theorem 8. What remains to be shown is that the bound on $\| \phi_{est,l} - \phi_l \|_{L^\infty[-1,1]}$ is at most a constant times the bound on $\| \tilde{\phi}'_l - \phi_l' \|_{L^\infty[-1,1]}$ for each $l \in S$. This is made precise in the following lemma.

Lemma 7. *Let α denote the error bounds of (D.4) or (D.6). We then have for all $l \in S$ that:*

$$\| \phi_{est,l} - \phi_l \|_{L^\infty[-1,1]} \leq 3\alpha$$

with $\phi_{est,l}$ as defined in (4.6).

Proof. We have that $\tilde{\phi}'_l(x) = \phi'_l(x) + p(x)$ for $x \in [-1, 1]$ with $\| p \|_{L^\infty[-1,1]} \leq \alpha$. This then gives us for any $x \in [-1, 1]$:

$$\int_{-1}^x \tilde{\phi}'_l(y)dy = \int_{-1}^x \phi'_l(y)dy + \int_{-1}^x p(y)dy, \tag{D.7}$$

$$\Rightarrow \psi_l(x) - \psi_l(-1) = \phi_l(x) - \phi_l(-1) + \int_{-1}^x p(y)dy, \tag{D.8}$$

$$\Rightarrow \int_{-1}^1 \psi_l(x)dx - \int_{-1}^1 \psi_l(-1)dx = \underbrace{\int_{-1}^1 \phi_l(x)dx}_{0} - \int_{-1}^1 \phi_l(-1)dx + \int_{-1}^1 \int_{-1}^x p(y)dydx, \tag{D.9}$$

$$\Rightarrow \int_{-1}^1 \psi_l(x)dx = 2(\psi_l(-1) - \phi_l(-1)) + \int_{-1}^1 \int_{-1}^x p(y)dydx. \tag{D.10}$$

By making use of (D.10) we thus have for any $x \in [-1, 1]$:

$$|\phi_{\text{est},l}(x) - \phi_l(x)| = \left| \psi_l(x) - \frac{1}{2}\int_{-1}^1 \psi_l(x)dx - \phi_l(x) \right|, \tag{D.11}$$

$$= \left| \psi_l(x) - \psi_l(-1) - (\phi_l(x) - \phi_l(-1)) - \frac{1}{2}\int_{-1}^1 \int_{-1}^x p(y)dydx \right|, \tag{D.12}$$

$$\leq \left| \int_{-1}^x (\tilde{\phi}'(y) - \phi'_l(y))dy \right| + \frac{1}{2}\int_{-1}^1 \int_{-1}^x |p(y)| \, dydx, \tag{D.13}$$

$$\leq \int_{-1}^x |p(y)| \, dy + \frac{1}{2}\int_{-1}^1 \int_{-1}^x |p(y)| \, dydx, \tag{D.14}$$

$$\leq 2\alpha + \frac{\alpha}{2}\int_{-1}^1 (1+x)dx = 3\alpha. \tag{D.15}$$

$\square$

Hence Lemma 7, Corollary 3 and Lemma 1 together complete the proof of Theorem 8.

E Proof of Theorem 3

Proof. Recall that the noisy linear system now has the form: $\mathbf{y}_i = \mathbf{V}\mathbf{x}_i + \mathbf{n}_i + \mathbf{z}_i$ where $z_{i,j} = (z'_{i,j} - z'_i)/\epsilon$ is the external noise component and $n_{i,j} = \frac{\epsilon}{2}\sum_{l \in S} v_{j,l}\phi''_l(\zeta_{i,j}^{(l)})v_{j,l}$ is the Taylor's remainder term for $i = -m_x, \ldots, m_x$ and $j = 1, \ldots, m_v$. We saw in the proof of Corollary 2 that $|n_{i,j}| \leq \frac{\epsilon k B_2}{2m_v}$. This gives us:

$$\| \mathbf{z}_i + \mathbf{n}_i \|_\infty \leq \left(\frac{2\kappa}{\epsilon} + \frac{\epsilon k B_2}{2m_v} \right) \text{ and } \| \mathbf{z}_i + \mathbf{n}_i \|_2 \leq \sqrt{m_v}\left(\frac{2\kappa}{\epsilon} + \frac{\epsilon k B_2}{2m_v} \right). \tag{E.1}$$

Recall that $\widehat{\mathbf{x}}_i$ is the solution returned by ℓ_1 minimization (B.1). By proceeding as in the proof of Corollary 2 we obtain for some constant $C > 0$ that

$$\| \widehat{\mathbf{x}}_i - \mathbf{x}_i \|_2 \leq \frac{2C\sqrt{m_v}\kappa}{\epsilon} + \frac{C\epsilon k B_2}{2\sqrt{m_v}} \tag{E.2}$$

holds with high probability for $i = m_x, \ldots, m_x$. As discussed in the proof of Lemma 4 in Section B, if we denote the R.H.S of (E.2) by τ then the thresholding procedure (B.6) with threshold τ recovers S exactly if $\tau < D/2$ holds. This is guaranteed if:

$$\frac{2C\sqrt{m_v}\kappa}{\epsilon} + \frac{C\epsilon k B_2}{2\sqrt{m_v}} < D/2 \tag{E.3}$$

$$\frac{CkB_2}{2\sqrt{m_v}}\epsilon^2 - \frac{D}{2}\epsilon + 2C\kappa\sqrt{m_v} < 0. \tag{E.4}$$

Clearly (E.4) has a solution if $\frac{D^2}{4} - 4C^2\kappa k B_2 > 0$ holds. In this case we then obtain the stated condition on ϵ as the solution range of (E.4). The rest follows in a straightforward manner as for the noiseless setting. $\square$

F Proof of Theorem 4

Proof. After resampling and averaging we have that:

$$\mathbf{z}_i = \left[\frac{(z'_{i,1} - z'_i)}{\epsilon} \; \frac{(z'_{i,2} - z'_i)}{\epsilon} \; \cdots \; \frac{(z'_{i,m_v} - z'_i)}{\epsilon} \right] ; i = -m_x, \ldots, m_x \tag{F.1}$$

where $z'_{i,j}, z'_i \sim \mathcal{N}(0, \frac{\sigma^2}{N})$ are i.i.d $\forall i, j$. We would like to ensure that $\left| z'_{i,j} - z'_i \right| < 2\kappa$ holds for any $\kappa > 0$ and $\forall i, j$ with high probability. Indeed we then obtain a bounded noise model and can simply use the analysis for the setting of arbitrary bounded noise from Section E.

Note that $z'_{i,j} - z'_i \sim \mathcal{N}(0, \frac{2\sigma^2}{N})$. It can be shown that for any $X \sim \mathcal{N}(0, 1)$ we have:

$$\mathbb{P}(|X| > t) \leq \frac{2e^{-t^2/2}}{t}, \quad \forall t > 0. \tag{F.2}$$

Since $z'_{i,j} - z'_i = \sigma\sqrt{\frac{2}{N}}X$ therefore for any $\kappa > 0$ we have that:

$$\mathbb{P}(\left| z'_{i,j} - z'_i \right| > 2\kappa) = \mathbb{P}\left(|X| > \frac{2\kappa}{\sigma}\sqrt{\frac{N}{2}} \right) \tag{F.3}$$

$$\leq \frac{\sigma}{\kappa}\sqrt{\frac{2}{N}} \exp\left(-\frac{\kappa^2 N}{\sigma^2} \right) \tag{F.4}$$

$$\leq \frac{\sqrt{2}\sigma}{\kappa} \exp\left(-\frac{\kappa^2 N}{\sigma^2} \right). \tag{F.5}$$

By taking a union bound over all $i = -m_x, \ldots, m_x$ and $j = 1, \ldots, m_v$ we have that

$$\mathbb{P}\left(\left| z'_{i,j} - z'_i \right| > 2\kappa : \forall i, j \right) \leq |\mathcal{X}| |\mathcal{V}| \frac{\sqrt{2}\sigma}{\kappa} \exp\left(-\frac{\kappa^2 N}{\sigma^2} \right). \tag{F.6}$$

For any $0 < p < 1$ we then have that $\mathbb{P}\left(\left| z'_{i,j} - z'_i \right| > 2\kappa : \forall i, j \right) < p$ if N satisfies:

$$N > \frac{\sigma^2}{\kappa^2} \log\left(\frac{\sqrt{2}\sigma}{\kappa p} |\mathcal{X}| |\mathcal{V}| \right). \tag{F.7}$$

This completes the proof. $\qquad\square$